

Unipolar Induction

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Abstract: The phenomenon of unipolar induction consists in generating electric field by uniformly rotating permanent magnets made of either conducting or dielectric substance. The origin of the field around conducting magnets is explained and calculated in Sect. 1. Dielectrics are considered in Sect. 2, where the role of the electric moment due to moving magnetic dipole moment is clarified. Sect. 3 is devoted to the description of the effect in the corotating frame of reference. In Sect. 4 the theory is applied to the Earth core and it is shown that the calculated electric field falls precisely into the range of the ambient electric field observed in the Tethered Satellite System mission of NASA. In the last paragraph I reflect on whether flyby anomaly might be attributed to an electric effect.

1 The phenomenon of unipolar induction

Consider a permanent magnet in the form of an electrically conducting disk magnetized along its symmetry axis and rotating around it with constant angular velocity Ω . If, using sliding contacts, one connects the axis and the rim of the disk by means of a rigid linear conductor electric current will go through the latter (unipolar generator, Fig. 1). An electromotive force arises even when the disk is made of a dielectric not supporting electric current. The mechanism which gives rise to this electromotive force is called *unipolar induction*².

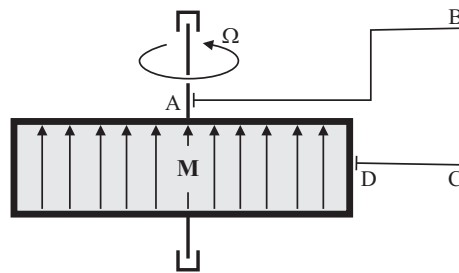


Figure 1

How to calculate the magnitude of the effect? When the disk is at rest and the conductor rotates in the opposite direction with angular velocity $-\Omega$, it

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²See e.g. J. Djurić *J. Phys. D.* **9**, 2623 (1976) and references to earlier works there.

crosses continuously the lines of force of the magnet and, therefore, an electromotive force

$$\mathcal{E}_{AD} = \int_{ABCD} [((-\boldsymbol{\Omega}) \times \mathbf{r}) \times \mathbf{B}] \cdot d\mathbf{r} \quad (1)$$

arises in it through motional induction. Since the *relative motion* is the same in both cases one might suppose that this same electromotive force arises also when the disk rotates together with the lines of force attached to it and the conductor remains at rest.

This reasoning is, however, of highly dubious value for at least two reasons.

First, *rotation is absolute* and it is by no means immaterial whether the disk or the conductor is rotating. In unipolar induction the rotation of the disk is supposed to take place in an inertial frame of reference. When, on the other hand, the disk is assumed resting, the point of view is changed and the reference frame becomes rotating. In mechanics a change of this kind is known to be accompanied by a corresponding change of the equations of motions, consisting in the inclusion of centrifugal and Coriolis forces. There is absolutely no reason to expect that Maxwell-equations remain intact under the same transformation (See Section 3 below).

Second, the lines of force are of a purely mathematical device, they motion can be neither observed nor calculated. It is, therefore, meaningless to differentiate, by reference to the motion of the lines of force, between the magnetic fields of magnets which rotate with different angular velocities around their symmetry axis. Hence in the two cases compared above the magnetic field is actually the same and the linear conductor which was originally at rest is simply replaced by a rotating one. These two situations are, however, completely unrelated since they differ from each other more than in a mere change of the point of view.

The correct explanation of the phenomenon is based on the observation, that the elements of the magnet rotate in the field of all the other elements. The magnet, therefore, rotates in its own magnetic field and — due to motional induction — an electromotive force

$$(\mathbf{V} \times \mathbf{B}) = [(\boldsymbol{\Omega} \times \mathbf{r}) \times \mathbf{B}]$$

will act in the element of it which at the given moment is in the position \mathbf{r} . Since the magnet is assumed conducting, the free charges get moved by this force until the field \mathbf{E} due to the volume and surface polarization charges compensates the induced electromotive force within the magnet:

$$\mathbf{E} = -(\mathbf{V} \times \mathbf{B}). \quad (2)$$

It is *the electric field of these polarization charges* outside the magnet which drives the current in the contour $ABCD$. Therefore, instead of (1), the correct electromotive force is given by the formula

$$\mathcal{E}_{AB} = \int_{ABCD} \mathbf{E} \cdot d\mathbf{r}. \quad (3)$$

The electric field, surrounding the magnet, remains obscured in the previous (erroneous) explanation³. But, in spite of their fundamental difference, *both the correct and the erroneous approaches lead to exactly the same electromotive force*. This follows from the general form⁴

$$\dot{\Psi} = - \oint_{C_t} [\mathbf{E} + (\mathbf{V} \times \mathbf{B})] \cdot d\mathbf{r} \quad (4)$$

of the law of induction which is valid even for moving contours C_t of continuously changing shape.

Assume that the contour $ABCD$ rotates together with the magnet and close it by adding a part DPA within the magnet. Due to axial symmetry the magnetic flux through this closed contour is constant in time, therefore

$$\oint_{ABCDPA} [\mathbf{E} + (\mathbf{V} \times \mathbf{B})] \cdot d\mathbf{r} = 0.$$

However, as a consequence of (2), the integrand on the part DPA vanishes and we have

$$\begin{aligned} \int_{ABCD} \mathbf{E} \cdot d\mathbf{r} &= - \int_{ABCD} (\mathbf{V} \times \mathbf{B}) \cdot d\mathbf{r} = \\ &= - \int_{ABCD} [(\boldsymbol{\Omega} \times \mathbf{r}) \times \mathbf{B}] \cdot d\mathbf{r} \end{aligned} \quad (5)$$

which proves the numerical equality⁵ of the two forms of \mathcal{E}_{AB} .

In general, however, the first method is by no means justified by this coincidence. When, for example, the magnet is a dielectric the first method leads obviously to the same electromotive force as before while the correct electromotive force given by the second method turns out to depend on the dielectric constant of the disk (see next section).

Summing up: Unipolar induction consists in the electric polarization of the permanent magnet, rotating around its symmetry axis. In the special case

³Since the polarization charges rotate together with the magnet they generate a constant magnetic field which modifies the original field of the magnet. It is easily seen that this secondary field is proportional to Ω^2 . In what follows we will confine ourselves to first order in Ω in which this modification of the magnetic field plays no role.

⁴The proof of this formula is found in Appendix 1.

⁵In §49 of L.D. Landau and E.M. Lifshitz *Electrodynamics of Continuous Media* (Pergamon Press 1984) unipolar induction is discussed in the rest frame of the magnet on the basis of the following remark:

It should be pointed out... that, if there is only one conductor moving as a whole (without change of shape) in an external magnetic field, then the solution of the problem is considerably simplified if we use a system of co-ordinates fixed in the conductor. In this system the conductor is at rest, and the external field varies with time in a given manner... This possibility does not depend on Galileo's (or on Einstein's) relativity principle, since the new system of co-ordinates is in general not inertial. The equivalence of the problems results from the above-mentioned fact that the electromagnetic induction is independent of the cause of the change in the magnetic flux.

of magnets made of conducting material the formula for the potential drop along contours like $ABCD$ coincides with the formula for the electromotive force induced in a contour rotating around a magnet which is at rest in an inertial frame of reference. In a practical calculation however, instead of (5), one can start from the known inside electric field (2) and calculate the outside field from the continuity of the tangential component of \mathbf{E} across the surface of the magnet and the absence of the net charge on it. For a spherical magnet of radius a and constant magnetic dipole density M this standard calculation leads to the potential

$$\Phi = -\frac{2}{9}a^5\mu_0M\Omega\frac{P_2(\cos\vartheta)}{r^3} \quad (r \geq a) \quad (6)$$

outside the sphere.

2 The role of the motionally induced electric dipole density

According to relativity theory the magnetic dipole density \mathbf{M} , moving with velocity \mathbf{V} , acquires electric dipole density \mathbf{P}_M . To linear order in Ω

$$\mathbf{P}_M = \frac{1}{c^2}(\mathbf{V} \times \mathbf{M}).$$

Does this electric polarization contribute to unipolar induction?

It does, but if the magnet conducts electricity, the consideration of the previous section remains unaltered. In stationary rotation the electric current density \mathbf{J} vanishes in both conductors and dielectrics. According to Ohm's law we have $\mathbf{J} = \sigma(\mathbf{E} + (\mathbf{V} \times \mathbf{B}))$. In conductors $\sigma \neq 0$, the condition $\mathbf{J} = 0$ leads to the electric field (2) inside the magnet and thereby the field outside the magnet becomes unambiguously determined. It is of no significance whether this field is generated solely by the free charges or the polarization charge density $-\text{div } \mathbf{P}_M$ also contributes to it.

When, on the other hand, the magnet is made of dielectric, in the Ohm's law we have $\sigma = 0$ and \mathbf{J} vanishes without requiring (2) to be fulfilled. In this case the relevant electrostatic problem can be solved, starting from the equation $\text{div } \mathbf{D} = 0$ where now

$$\mathbf{D} = \varepsilon\mathbf{E} - \varepsilon_0\chi(\mathbf{V} \times \mathbf{B}) - \mathbf{P}_M.$$

The second term on r.h.s. is the polarization generated by the motionally induced electromotive force (χ is the electric susceptibility).

We arrive again at a well defined electrostatic problem which is easily solved for a spherical magnet. For the potential difference between the axis of rotation and the "equator" the formula

$$\Delta\Phi^{ins} = \Delta\Phi^{cond} \cdot \frac{2\varepsilon + \varepsilon_0}{2\varepsilon + 3\varepsilon_0}$$

is obtained (Φ^{cond} is given in (6)). As $\varepsilon \rightarrow \infty$ we have $\Delta\Phi^{ins} \rightarrow \Delta\Phi^{cond}$ as expected. When the magnet is neither conducting nor polarizable we obtain $\Delta\Phi_{\varepsilon=\varepsilon_0}^{ins} = \frac{3}{5}\Delta\Phi^{cond}$ which is the contribution of \mathbf{P}_M alone to unipolar induction.

3 Transition to the corotating frame

Reference frame must always be chosen so as to simplify the analysis of the problem under study. Since there are important problems which are most conveniently studied in the rest frame of the rotating magnet we turn to the description of the unipolar induction in this latter frame. Observable properties such as whether an electric bulb in the contour $ABCD$ of Fig. 1 is or is not gleaming cannot, of course, be altered by mere change of the point of view but to verify this we must first transcribe Maxwell-equations into rotating frame.

In order to anticipate the necessary modifications assume that in the immediate vicinity of the axis of rotation a constant linear charge density of magnitude λ is concentrated. Then, if the equation $\text{div } \varepsilon_0 \mathbf{E} = \rho$ remained valid in the rotating frame too, the flux of \mathbf{E} through the multitude of cylindrical surfaces, surrounding the axis of rotation, would remain equal to λ/ε_0 per unit height in both rotating and inertial frames and this requirement would strongly limit the possible change of the electric field in the transition from an inertial frame to the corotating one.

But such a change must take place. The Coulomb force $e\mathbf{E}$ by which the charge density λ acts on a point charge e is certainly different when the charge is at rest with respect to the one or the other frame. This is suggested by the transformation law

$$\mathbf{E}' \approx \mathbf{E} + (\mathbf{V} \times \mathbf{B}) \quad (V \ll c)$$

between two inertial systems. Though inapplicable when transition takes place to rotating frame, this equation suggests that the electric field does change in the latter transition also, leading an observable difference in the value of the Coulomb force in the two frames.

Let us assume now that an electric current I flows along the axis of rotation. Then, if the equation $\text{rot } \mathbf{B} = \mu_0 \mathbf{J}$ remained valid in the rotating frame, the field \mathbf{B} would presumably also have to remain essentially the same. But this would almost certainly be in conflict with the relativistic transformation properties of the fields and the expected modification of the Lorentz-force.

The way out of this dilemma is suggested by the electrostatics of polarized media: On the level of field equations the fields \mathbf{E} , \mathbf{B} which determine the force exerted on charged particles must be distinguished from the fields \mathbf{D} , \mathbf{H} generated by the latter. At the same time, an appropriate pair of "material equations" must be postulated in order to restore the correct number of degrees of freedom. The whole process must naturally be based on the known relativistic transformation properties of the field in the form of an antisymmetric field tensor F^{ij} .

The shortest way to this form of electrodynamics in a rotating frame goes through general relativity where the problem has been solved in complete generality for an arbitrary system of coordinates⁶. Specification of the general formulae leads to the following equations valid in rotating coordinate system:

$$\begin{aligned} \operatorname{div} \mathbf{B} = 0 \quad \frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{E} &= 0 \\ \operatorname{div} \mathbf{D} = \rho \quad \frac{\partial \mathbf{D}}{\partial t} &= \operatorname{rot} \mathbf{H} - \mathbf{J}. \end{aligned} \quad (7)$$

The "material equations" are

$$\mathbf{D} = \frac{\varepsilon_0}{\sqrt{1 - V^2/c^2}} \mathbf{E} + \frac{1}{c^2} (\mathbf{H} \times \mathbf{V}) \quad (8)$$

$$\mathbf{B} = \frac{\mu_0}{\sqrt{1 - V^2/c^2}} \mathbf{H} + \frac{1}{c^2} (\mathbf{V} \times \mathbf{E}). \quad (9)$$

$\mathbf{V}(\mathbf{r})$ is a vector field present in the rotating frame given by the formula $\mathbf{V} = (\mathbf{r} \times \boldsymbol{\Omega})$. In the rotating frame Lorentz-force acquires an additional term:

$$\mathbf{F}_L = e\mathbf{E} + e(\mathbf{v} \times \mathbf{B}) - \frac{e}{c^2} (\mathbf{E} \cdot \mathbf{v}) \mathbf{V}. \quad (10)$$

Apply now these equations to unipolar induction. To this end we have to replace \mathbf{H} in (9) by the sum $\mathbf{H} + \mathbf{M}$ within the magnet. The jump conditions on the surface of the magnet follow from (7) and are the usual ones: Normal components of \mathbf{D} and \mathbf{B} and tangential components of \mathbf{E} and \mathbf{H} are continuous which implies the continuity of the electric potential.

The same argument which in Section 1 led to the relation (2) now gives $\mathbf{E} = 0$ inside the magnet. Outside we have $\operatorname{div} \mathbf{D} = \rho = 0$ which leads to the equation

$$\operatorname{div} \left(\frac{\varepsilon_0}{\sqrt{1 - V^2/c^2}} \mathbf{E} \right) = -\frac{1}{c^2} \operatorname{div}(\mathbf{H} \times \mathbf{V}). \quad (11)$$

This formula is the starting point of the calculation of the electric field in the corotating frame.

As we have mentioned earlier⁴ our considerations in the inertial frame were implicitly restricted to linear order in Ω since back reaction of the induced currents on the magnetic field was neglected. In the rotating frame no such currents arise and nonlinearity in Ω is made explicit in the equations (7)-(9) which is a definite advantage of the rotating frame.

In order to confine ourselves to linear order again⁷ we note that when $\Omega = 0$ the electric field \mathbf{E} obviously disappears and so it is proportional to Ω . Since

⁶See L.D. Landau and E.M. Lifshitz *The Classical Theory of Fields* Fourth Revised English Edition, § 90, p. 275, and Appendix 2 at the end of the present work.

⁷This limit can be directly contrived without making the detour through general relativity (see Landau and Lifshitz *Electrodynamics of Continuous Media* § 76).

the same is true for the field $\mathbf{V}(\mathbf{r})$ the second term on the r.h.s. of (9) is of the order of Ω^2 and must be neglected. At the same time the square roots in the "material equations" and in (11) must be replaced by 1.

In this approximation, therefore, the magnetic field in the corotating system is exactly the same well known dipole field as in the laboratory (inertial) frame⁸. It can be substituted into the r.h.s. of (11) which can then be used to calculate \mathbf{E} . Since the fields are static we have from (7) $\text{rot } \mathbf{E} = 0$ and an electric potential Φ' may be introduced into (11) by the usual relation $\mathbf{E} = -\text{grad } \Phi'$:

$$\Delta\Phi' = \frac{1}{c^2} \text{div}(\mathbf{H} \times \mathbf{V}). \quad (12)$$

This equation must be solved under the following conditions: (1) Since the tangential component of the electric field on the surface of the magnet is continuous and, therefore, equal to zero Φ' must be constant along this surface; (2) Since the magnet is uncharged the flux of the field \mathbf{D} across the closed surfaces, surrounding the magnet, must be zero. For the spherical magnet the solution is

$$\Phi' = \Phi - \frac{1}{3}a^3\mu_0M\Omega \frac{\sin^2\vartheta}{r} \quad (r \geq a) \quad (13)$$

which is to be compared with (6).

Let us calculate finally the electromotive force \mathcal{E}'_{AB} in the contour $ABCD$ of Section 1 which rotates with angular velocity $\omega = -\Omega$ in the corotating system. This force is determined by the Lorentz-force equation (10) in which now $\mathbf{v} = (\boldsymbol{\omega} \times \mathbf{r}) = -(\boldsymbol{\Omega} \times \mathbf{r})$. The third term of the force is of the order of Ω^2 and must be neglected while the first term's contribution vanishes since the endpoints of the contour lie on the surface of the magnet which is equipotential. We have, therefore

$$\mathcal{E}'_{AB} = \int_{ABCD} [((-\boldsymbol{\Omega}) \times (\mathbf{r})) \times \mathbf{B}] \cdot d\mathbf{r}$$

which is exactly the same expression as (1) and (5). When the magnet is insulating its surface ceases to be equipotential and the first term of (10) also contributes to the electromotive force. This contribution is determined directly by $\boldsymbol{\Omega}$, being independent of \mathbf{v} .

4 Application of the theory to the Earth

Erth's core is a huge rotating magnet and it is tempting to apply the ideas of the preceding sections to it. To start with let us assume that (1) the core is a permanent magnet with nonzero conductivity⁹, (2) the magnetic field outside

⁸In the literature on the magnetic hydrodynamics of rotating fluids this fact is often expressed as e.g. "rigid body rotation... rotates a magnetic field without distortion" (H. K. Moffat *Magnetic Field Generation in Electrically Conducting Fluids* Cambridge 1978, p. 53)

⁹The Glatzmaier-Roberts model of the geomagnetic field (G.A.Glatzmaier and P.H.Roberts *Contemporary Physics* **38**, 269 (1997)) would certainly make possible a much more realistic treatment.

the core is a pure dipole field, (3) the geographical and magnetic poles coincide, (4) the mantle is electrically neutral ($\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $\sigma = 0$)¹⁰, and finally (5) the atmosphere is absent.

These are mostly very crude assumptions but they permit us to apply the formulas of the previous sections without further refinements. The electric field at a given point on the surface of the Earth can be calculated from (13). Assumption (2) allows to replace the unknown polarization density M with the empirical value B of the magnetic field on the Equator, equal to $B = 33 \mu T$. On the Equator the electric field given by (13) is a purely radial one and is given by the formula

$$E' = \Omega R_c \left(\frac{R}{R_c} - \frac{R_c}{R} \right) \cdot B$$

where R and R_c are the Earth's and the core's radiuses respectively. Substituting $R_c \approx R/2 \approx 3.10^6 m$, $\Omega = 2\pi/24 h^{-1} = 7.3 \cdot 10^{-5} s^{-1}$ and $B = 33.10^{-6} T$ we obtain $E \approx 10 mV/m$ which is in itself of a measurable magnitude but stronger fields of atmospheric origin make its observation rather problematic.

In march of 1996 a spherical 1.6-meter diameter satellite was released out from the payload bay of Space Shuttle Columbia during its orbiting at a height about $h=90$ km above the Earth. Its tether a long conducting cable (up to the length of 19.1 km) of resistance R , served (among others) to measure the radial electric field (called *ambient field*) in the vicinity of the orbit. The system is conveniently called Tethered Satellite System (TSS).

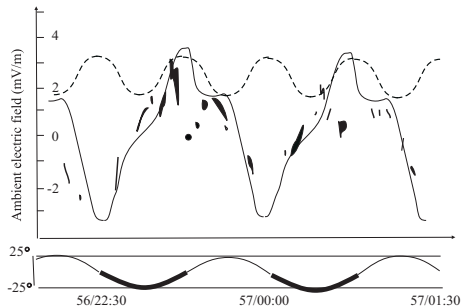


Figure 2

Free electrons in the ionosphere where TSS operated were attracted to the satellite. The electrons travelled along the tether to the orbiter, producing the current I . The electric circuit was closed by means of an electron generator on the orbiter which returned charged particles back into the ionosphere. The product IR gave the electromotive force along the tether the dominant part of which was due to the motional induction in the magnetic field of the Earth¹¹

¹⁰Electric polarizability of the mantle makes the electric field larger. When the radius of the core is half of that of the Earth, the correction factor is equal to $(3301\varepsilon_r - 2321)/(670\varepsilon_r + 310)$ where ε_r is the global relative electric permittivity of the mantle.

¹¹Since this electromotive force is of outward direction (opposite to the Earth) the orbiter had to revolve nearer to the Earth.

and could be calculated with sufficiently high accuracy. From the rest of the electromotive force the ambient field could be estimated.

Results of the measurement¹² are summarized on Fig. 2. The solid line is a theoretical curve calculated from the theory of "tidally driven neutral winds" which is outside my competence. The agreement with observations is considered satisfactory.

The dashed curve added by myself has been calculated according to the formula

$$E_r(t) = -2(\Omega R_c) \frac{R_c}{R} P_2(-\sin \alpha \cdot \sin \omega t) \cdot B$$

derived from (6); α denotes the angle between the orbital and equatorial planes and ω is the angular velocity of the satellite system (the radius $R + h$ of the orbit was replaced by R). In the actual experiment their values were $\alpha = 25^\circ$ and $\omega = 2\pi/90 \text{ min}^{-1}$. The azimuthal angle ωt in the orbital plane is measured from the intersection with the equatorial plane.

Though the order of magnitude agreement is quite impressive it must not be taken at face value. The curve was calculated under the assumption (5) of no atmosphere while the measurement itself was made possible exclusively thanks to the existence of the ionosphere. What the order of magnitude of the electric field does probably indicate is that this field should be taken into consideration among the *input data* when complicated processes in the ionosphere are studied.

One may wonder whether unipolar induction in the Earth's core might not have some unexpected influence on satellites not specifically designed to observe the ambient electric field. Flyby anomaly¹³ is a phenomenon which, in spite of the efforts of its discoverers, remained so far unexplained. In this situation even the most improbable explanations deserve attention. A suggestion of this kind might be an effect due to the electric field of the Earth. If satellites can somehow (perhaps by photoeffect) get electrically charged then the quadrupole character of this field might account for the latitude dependence of the observed anomaly.

Appendix 1: Derivation of (4)¹⁴

Let us start with formula

$$\Psi(t) = \int_{\Sigma} \mathbf{B} \cdot d\Sigma = \oint_{C_t} \mathbf{A} \cdot d\mathbf{r} \quad (14)$$

for the magnetic flux Ψ through the closed material contour C_t , following from the definition $\mathbf{B} = \text{rot } \mathbf{A}$ of the vector potential. The contour C_t may vary its shape and length in time and is given in parametric form as

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(\lambda, t) & (0 \leq \lambda \leq 1), \\ \mathbf{r}(1, t) &= \mathbf{r}(0, t), \end{aligned}$$

¹²S. D. Williams et al. *Geophys. Res. Lett.*, **25** 445 (1998). The lower curve shows TSS latitude with thick curve indicating local night.

¹³J.D.Anderson et al, *Phys. Rev. Lett.* **100**, 091102 (2008).

¹⁴H. K. Moffat *op. cit.* p. 32

where it is understood that, during the deformations, the same value of the parameter λ remains attached to the same material point¹⁵.

Since the contour is closed the vector function $\mathbf{r}(\lambda, t)$ is a periodic function of λ of period 1: $\mathbf{r}(\lambda + 1, t) = \mathbf{r}(\lambda, t)$. The velocity of the point at λ is equal to $\mathbf{v} \equiv \dot{\mathbf{r}} = \frac{\partial \mathbf{r}(\lambda, t)}{\partial t}$.

Using this representation of the contour, the integral in (14) can be written in the form

$$\Psi(t) = \int_0^1 A_i \frac{\partial x_i(\lambda, t)}{\partial \lambda} d\lambda,$$

where the notation $x_1, x_2, x_3 \equiv x, y, z$ is introduced and summation over repeated indices is understood. In the integrand the components $A_i \equiv A_i(\mathbf{r}(\lambda, t), t)$ are functions of both t and λ .

For the time derivative of the flux we obtain

$$\begin{aligned} \frac{d\Psi}{dt} &= \int_0^1 \frac{d}{dt} \left(A_i \frac{\partial x_i}{\partial \lambda} \right) d\lambda = \\ &= \int_0^1 \left[\frac{dA_i}{dt} \cdot \frac{\partial x_i}{\partial \lambda} + A_i \frac{\partial}{\partial t} \left(\frac{\partial x_i}{\partial \lambda} \right) \right] d\lambda. \end{aligned} \quad (15)$$

Here

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j} \frac{\partial x_j}{\partial t} = \frac{\partial A_i}{\partial t} + v_j \frac{\partial A_i}{\partial x_j} \quad (16)$$

is the "material derivative" of the vector potential on the contour.

In the second term of the integrand

$$\frac{\partial}{\partial t} \left(\frac{\partial x_i}{\partial \lambda} \right) = \frac{\partial}{\partial \lambda} \left(\frac{\partial x_i}{\partial t} \right) = \frac{\partial v_i}{\partial \lambda},$$

hence

$$\frac{d\Psi}{dt} = \int_0^1 \left(\frac{\partial A_i}{\partial t} \frac{\partial x_i}{\partial \lambda} + v_j \frac{\partial A_i}{\partial x_j} \frac{\partial x_i}{\partial \lambda} + A_i \frac{\partial v_i}{\partial \lambda} \right) d\lambda. \quad (17)$$

The third term can be written as

$$\frac{\partial}{\partial \lambda} (A_i v_i) - \frac{\partial A_i}{\partial \lambda} v_i,$$

the first of which does not contribute:

$$\int_0^1 \frac{\partial}{\partial \lambda} (A_i v_i) d\lambda = (A_i v_i)_{\lambda=1} - (A_i v_i)_{\lambda=0} = 0,$$

and so in the integrand of (17) we have

$$A_i \frac{\partial v_i}{\partial \lambda} = -v_j \frac{\partial A_j}{\partial \lambda} = -v_j \frac{\partial A_j}{\partial x_i} \frac{\partial x_i}{\partial \lambda}.$$

¹⁵Since the integral on the r.h.s. of (4) is not invariant under time dependent reparametrization of the contour it is defined only if the parametrization is given in physical terms (up to an arbitrary choice at some given moment of time).

The formula (17) can then be transformed into

$$\frac{d\Psi}{dt} = \oint_{C_t} \left[\frac{\partial A_i}{\partial t} + v_j \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) \right] dx_i. \quad (18)$$

In the integrand

$$v_j \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) = -(\mathbf{v} \times \text{rot } \mathbf{A})_i,$$

therefore

$$\frac{d\Psi}{dt} = \oint_{C_t} \left[\frac{\partial \mathbf{A}}{\partial t} - (\mathbf{v} \times \text{rot } \mathbf{A}) \right] \cdot d\mathbf{r}. \quad (19)$$

Here

$$\text{rot } \mathbf{A} = \mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} - \text{grad } \phi,$$

where ϕ is the scalar potential. Substituting these into (19) the scalar potential gives no contribution and we arrive at the formula (4).

This formula, therefore, is an exact consequence of the Maxwell-equations alone but it becomes especially useful when, taking into consideration the Lorentz-force too, we interpret the integral in it as the total electromotive force, acting in the contour. This interpretation, however, is valid only in quasistationary approximation i. e. when the perturbances may be assumed to cross the domain of the contour instantaneously.

Appendix 2: Maxwell-equations in general coordinates

Here we summarize the formulae of Landau and Lifshitz⁶, using SI.

The space-time coordinate system is characterized by the metric tensor g_{ij} (latin indices take on values 0, 1, 2, 3). The three dimensional metric is denoted by $\gamma_{\alpha\beta}$ (greek indices are of value 1, 2, 3) which in terms of g_{ij} is given by the formula

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + hg_{\alpha}g_{\beta}, \quad (20)$$

where

$$h = g_{00}, \quad g_{\alpha} = -\frac{g_{0\alpha}}{g_{00}}.$$

Further quantities which will be needed are

$$\begin{aligned} \gamma &= \det \gamma_{\alpha\beta} & g &= \det g_{ij} = -h\gamma \\ \gamma^{\alpha\beta} &= -g^{\alpha\beta} & g^{\alpha} &= \gamma^{\alpha\beta} g_{\beta} = -g^{0\alpha}. \end{aligned}$$

We have further

$$g_{ik}g^{kj} = \delta_i^j, \quad \gamma_{\alpha\sigma}\gamma^{\sigma\beta} = \delta_{\alpha}^{\beta}.$$

The Maxwell-equations are

$$\text{div } \mathbf{B} = 0 \quad \text{rot } \mathbf{E} = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma} \mathbf{B}) \quad (21)$$

$$\text{div } \mathbf{D} = 0 \quad \text{rot } \mathbf{H} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma} \mathbf{D}) + \mathbf{J}, \quad (22)$$

in which rot and div are understood in form valid in general coordinates:

$$\begin{aligned} (\text{rot } \mathbf{a})^\alpha &= \frac{1}{2\sqrt{\gamma}} \epsilon^{\alpha\beta\gamma} \left(\frac{\partial a_\gamma}{\partial x^\beta} - \frac{\partial a_\beta}{\partial x^\gamma} \right), \\ \text{div } \mathbf{a} &= \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} (\sqrt{\gamma} a^\alpha) \end{aligned} \quad (23)$$

The "material equations" are

$$\begin{aligned} \mathbf{D} &= \frac{\epsilon_0}{\sqrt{h}} \mathbf{E} + \frac{1}{c} (\mathbf{H} \times \mathbf{g}) \\ \mathbf{B} &= \frac{\mu_0}{\sqrt{h}} \mathbf{H} + \frac{1}{c} (\mathbf{g} \times \mathbf{E}). \end{aligned} \quad (24)$$

The continuity equation:

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma} \rho) + \text{div } \mathbf{J} = 0, \quad (25)$$

and, finally, the Lorentz-force:

$$\mathbf{F}_L = e\mathbf{E} + e(\mathbf{v} \times \mathbf{B}) - \frac{e}{c} (\mathbf{E} \cdot \mathbf{v}) \mathbf{g}.$$

The connection between 3D and 4D quantities:

$$\begin{aligned} E_\alpha &= cF_{0\alpha} & B^\alpha &= -\frac{1}{2\sqrt{\gamma}} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \\ D^\alpha &= -\epsilon_0 c \sqrt{h} F^{0\alpha} & \mu_0 H_\alpha &= -\frac{\sqrt{-g}}{2} \epsilon_{\alpha\beta\gamma} F^{\alpha\beta}. \end{aligned}$$

From these the components E^α , B_α , D_α and H^α are obtained with the aid of the 3-dimensional metric.

Specialization to rotating coordinates: In cylindrical coordinates the transformation formulae from the (primed) inertial system to the (unprimed) rotating system are

$$t' = t, \quad r' = r, \quad \vartheta' = \vartheta, \quad \varphi' = \varphi + \Omega t$$

which leads to the line element

$$\begin{aligned} ds^2 &= \left[1 - \left(\frac{\Omega r \sin \vartheta}{c} \right)^2 \right] c^2 dt^2 - 2 \frac{\Omega r^2 \sin^2 \vartheta}{c} d\varphi d(ct) - \\ &\quad - dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \end{aligned}$$

From this line element the metric tensor g_{ij} (and h) can be read off and we have

$$\begin{aligned} g_r &= g_\vartheta = 0 & g_\varphi &= \frac{\Omega r^2 \sin^2 \vartheta}{c h} \\ \gamma_{rr} &= 1, & \gamma_{\vartheta\vartheta} &= r^2, & \gamma_{\varphi\varphi} &= \frac{r^2 \sin^2 \vartheta}{h}. \end{aligned}$$

Since the gamma-tensor is diagonal we have $\gamma^{\alpha\alpha} = 1/\gamma_{\alpha\alpha}$ and, therefore

$$g^r = g^\vartheta = 0, \quad g^\varphi = \frac{\Omega}{c}.$$

These formulae lead immediately to the equations (7) -(10) in which the space coordinates can be chosen arbitrarily.