Time and Relativity

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I plan to follow essentially (though not exactly) the same line of thought that I did last time, in the lectures two years ago.

The theory of relativity consists of two clearly distinguishable parts: special relativity and general relativity. According to common opinion, the special theory is relatively simple, whereas the general theory is very difficult. I used to think so myself, but by now I am convinced that this is true only of the mathematical apparatus used in the two theories: to understand special relativity, high-school mathematics is entirely sufficient (perhaps including some elements of differential and integral calculus), whereas the mathematical apparatus of general relativity is difficult (but beautiful). Conceptually, however, the gap between Newtonian physics and special relativity is in fact much wider and deeper than the one that separates the special theory from the general one, and this is what causes the real difficulty.

I am convinced that the key to relativity theory lies in achieving the clearest possible understanding of the Einsteinian concept of time, and therefore we will base the entire line of reasoning on this.

1. Two Kinds of Time

The core of special relativity consists in a careful analysis of the concept of physical time. What is at issue is the uncovering of properties of time that do not at all present themselves to everyday thinking. And since Newtonian physics adopted this unreflected, commonplace concept of time, these properties remained unnoticed in classical physics as well.

What is it all about? When we travel by train, the time that elapses between two stations can be determined most simply by looking at our wristwatch, which we assume to be ideal[1]. But if we do not have our own watch, it is enough to look at the clocks in the railway stations, from which we can read off when we arrived at a given town, and by a simple subtraction we can likewise obtain how much time elapsed between two stations.

One might think that the two procedures are completely equivalent, yet there is a rather essential difference between them—even if we also assume the station clocks to be ideal. For between two stations the same amount of time elapses according to our wristwatch and according to the station clocks *only if the latter are correctly synchronized with one another*.

[1] By an ideal clock, we mean the hypothetical endpoint of the developmental chain: sandglass and water clock → escapement-driven wheel clock → pendulum clock → chronometer → quartz clock → atomic clock → …

How can this be achieved? In practice, by using the precise time signal of the radio; but in principle, since our clocks are ideal, we may also assume that all the clocks of all the railway stations in the country were manufactured in a single clock factory, and when they were all finished, their hands were set in a common room at exactly noon to the same “12 o’clock” position. After that they could calmly be delivered to their final locations, because—being ideal—they would preserve the correct time. But it is also possible that someone went around to all the railway stations and set the clocks according to his own watch.

Let us observe the difference carefully! If we read off the same time interval from a single clock, no synchronization of any kind is necessary; it is sufficient that the clock be accurate (ideal). In the case of readings taken from several clocks, however, they must also be synchronized.

But what makes this difference unavoidable? The answer is obvious: our wristwatch was able to cope with the task on its own only because it was moving, whereas the station clocks are continuously at rest in one place.

We shall now immediately examine what consequences all this has for the tasks that arise in physics, but first, so that we do not have to describe at length in every case which procedure is being used, let us give names to the two time intervals obtained by the two procedures.

If the time interval between two given instants (events) is read off from a single clock, we shall call the elapsed interval *proper time* and denote it by $\Delta\tau$. In the other case, when the time between the same two events is determined on the basis of two different, correctly synchronized clocks at rest at the locations of the events, we shall use the name *coordinate time* and the notation Δt.

Let us now consider what follows from all this for physics—more specifically, for kinematics. Consider two point masses moving along the x-axis and represent their motion in the usual way (Fig. 1):

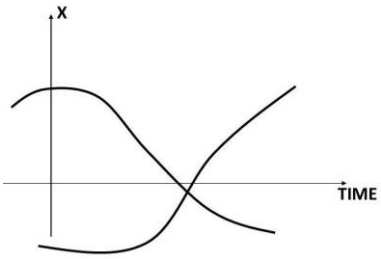

Figure 1

Which type of time does the time plotted on the horizontal axis belong to?

To answer this, it is again worth turning to railway transportation, because a timetable is essentially a collection of such graphs (Figure 2).

On the horizontal axis, we obviously understand the time shown by the clocks at the stations, that is, the coordinate time, not the position of the hands on the locomotive drivers’ watches (their proper time).

The name *t* is coordinate time because it is one of the coordinates, just like *X, Y, Z*.

It is no different in physics. It is often necessary to represent the trajectories of several mass points simultaneously on a single graph; *conceptually*, the coordinate time is what they have in common, therefore on the time axis we must plot the coordinate time[2].

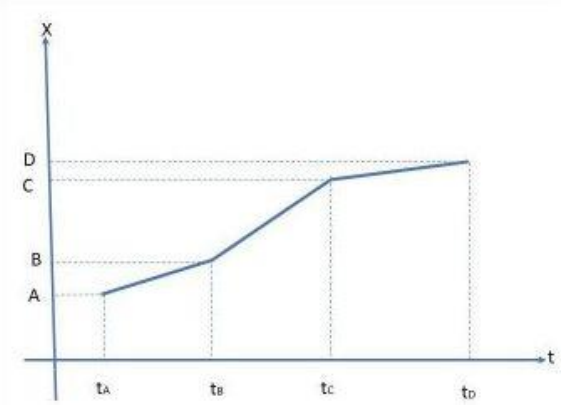

Figure 2

To give a more precise definition of coordinate time, it is worth considering the following example. Suppose that the temperature is continuously measured at the stations. If we are interested in what the temperature was at different stations at the same moment in time (say at 12 noon), it is completely obvious that the "time" here can only mean coordinate time: the time shown by clocks at the stations, at rest near the thermometers.

In mathematical terms, the temperatures being measured here form a *temperature field T(**r**,t)*. The general lesson from this example is that for fields—which are functions of position and time—the time always automatically means coordinate time.

This naturally also applies to the electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{B}(\boldsymbol{r},t)$, whose bracket always contains the coordinate time t, and to Maxwell's equations, of which we write just one as an example:

$$\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} + rot\boldsymbol{E}(\boldsymbol{r},t) = 0. \qquad (1)$$

But to understand all this correctly, we obviously need to go beyond illustrating coordinate time with clocks at railway stations, and we must provide the general definition of coordinate time.

First of all, we must establish that clocks indicating coordinate time—unlike the clocks at railway stations—are in fact only *imagined*; in general, these clocks "are not there."
Briefly, we express this by saying that they are *virtual* (though we allow that, similarly to station clocks, we may realize as many of them as necessary).

[2] It would not be a good justification to say that we have to choose coordinate time because mass points do not have clocks. In principle, we can attach a tiny ideal clock to every mass point, which would show the proper time of the mass point.

After this, a precise physical definition of coordinate time can be formulated as follows:

Coordinate time would be indicated by densely distributed, correctly synchronized ideal clocks at rest in space, if they were actually present there.

Such a general definition of coordinate time is in fact required not only for the description of fields, but also for the description of the motion of mass points. The trajectory of a mass point is given by the equations

$$x = f(t), \qquad y = g(t), \qquad z = h(t) \tag{2}$$

This is to be understood as follows: when the virtual clock located at the position of the mass point shows, say, *t = 5* seconds, then the mass point is located precisely at the point az $x = f(5), y = g(5), z = h(5)$. In the framed definition of coordinate time, the qualifier "correctly synchronized" naturally requires explanation. This most essential element of the definition has a different meaning in Newtonian physics than it does in the theory of relativity.

In Newtonian physics, "correct synchronization" means that clocks indicating coordinate time must show the same absolute time everywhere (the choice of the origin and the unit of time is, of course, arbitrary). As to what is meant by this, we may read the following laconic statement in the *Principia*:

Absolute, true, and mathematical time, of itself and from its own nature, flows equably without relation to anything external …

From this definition it does not at all become clear how (or even whether)—according to the Newtonian assumption—clocks indicating absolute time are to be set (synchronized) in such a way that the origin of timekeeping is the same everywhere. But had someone asked about this, Newton himself and his successors would surely have said that it is a very simple matter: two (ideal) clocks are synchronized by bringing them to the same place, setting their hands to the same time, and then carrying them to where they are needed. Since they are ideal, they keep time at the rhythm of absolute time even while being transported, and they do not lose their synchrony. One may in fact convince oneself of this by later bringing them together again at one location and observing that they still show exactly the same time. A collection of clocks may be synchronized with one another in some order, pairwise or by gathering them into a group.

Absolute time may otherwise be imagined as an imperfect GPS that continuously transmits time signals every second, but without attaching time stamps to them.

To my knowledge, before Einstein no one explicitly discussed this synchronization procedure; but I am convinced that this was simply because it was regarded as obvious. Einstein was the first to realize that *a problem may arise* here.

The theory of relativity provides a radically different, yet equally unambiguous, recipe for what is meant by the correct synchronization of virtual clocks indicating coordinate time. However, in order to elucidate this, we must also examine the problem from another direction, starting from electrodynamics. We shall do this in the next chapter. As a conclusion to this chapter, we now summarize once again what we mean precisely by the two kinds of time:

1) What a clock shows is its own *proper time*. A mass point also has a proper time, which would be shown by an ideal clock imagined to be attached to it. The proper time elapsed at

individual points on the Earth's surface is shown by an ideal clock (real or merely imagined) that is at rest at that point.

If a proper-time interval $\Delta\tau$ elapses between two events on a given clock (or on a given mass point), then this is a fact that remains unchanged from every point of view. Expressed differently: the proper-time interval is invariant.

2) The clocks that indicate coordinate time also show their "own proper time," but the essence of coordinate time is that these individual proper times are fitted into some system time, which is the consequence of an appropriate synchronization of the clocks. The coordinate-time interval Δt that elapses between two instantaneous events occurring at different locations depends on the concrete method of synchronization and therefore is not necessarily invariant.

In Newtonian physics, no distinction was made between the coordinate-time interval and the proper-time interval. Let the mass point again move along the trajectory

$$x = f(t), \qquad y = g(t), \qquad z = h(t).$$

How much proper time elapses on the mass point during the coordinate-time interval Δt=3 seconds, while the mass point moves from the point $x = f(5), y = g(5), z = h(5)$ to the point $x = f(8), y = g(8), z = h(8)$? According to Newtonian physics (and the naïve, general concept of time), obviously also 3 seconds. In general:

$$\Delta\tau = \Delta t. \qquad (newtonian) \tag{3}$$

In Newtonian synchronization, we exploit the fact that the operation of ideal clocks is not affected even by their transport (if this were not the case, then when the clocks were brought together again at one location, they would no longer be synchronized).

This equality is so deeply embedded in our way of thinking that we do not even notice it: the proper-time interval and the coordinate-time interval are durations *defined in two different ways*, and therefore it does not at all follow from their very concepts that—when both are meaningful—they must be equal to each other.

The essential element of the theory of relativity is the clear distinction between these two kinds of time concepts and the consistent exploitation of the possibilities arising from their difference.

2. First postulate of relativity[3]

In a June 1905 paper, Einstein published the theory of relativity in an almost fully developed form. Let us begin this chapter with the example that Einstein used to introduce this work in the very first paragraph of his paper.

In modern terminology, he compared motional induction and induction at rest.

Imagine a bar magnet standing on a table and a coil nearby (connected to an ammeter).

First, consider the case when the magnet is at rest and the coil moves with constant velocity $\boldsymbol{V}$. In the magnetic field $\boldsymbol{B}$ of the bar magnet, the Lorentz force $\boldsymbol{F} = Q(\boldsymbol{V} \times \boldsymbol{B})$ acts on every conduction electron (with charge *Q*) in the coil, producing a (variable) current indicated by the ammeter. This

[3] Traditionally, this is usually taken as the second.

phenomenon is *motion-induced (or motional) induction*. Einstein emphasizes that during this process, no electric field is generated anywhere.

Now assume that the coil is at rest and the magnet moves parallel to itself with velocity -***V***. No Lorentz force acts now on the coil, since it is at rest. However, the magnetic flux Ψ through the coil changes, and therefore an electromotive force $\mathcal{E} = -\dot{\Psi}$ is generated in it, producing a current, and the ammeter likewise deflects. This phenomenon is *induction at rest*, which—as Einstein also stresses—is due to the presence of an electric field.

According to experience, the deflection of the ammeter is the same in both cases. The explanation of this coincidence is straightforward: the two situations differ only in that the same phenomenon is first observed from the rest frame of the magnet and second from the rest frame of the coil (both of which can, of course, be considered inertial frames). Since the pointer position of the galvanometer is an objective fact that cannot depend on the coordinate system chosen for the description (it is invariant), it is *evident* that in the two—otherwise indeed different—phenomena, the same variable current must be measured.

However, this explanation leads to an unacceptable consequence.

For if it were really true that the laws of electromagnetism are equally valid in all inertial frames, then Maxwell's equations would hold in every inertial frame. However, from Maxwell's equations it follows that light propagates in all directions with the same speed $c = 1/\sqrt{\varepsilon_0 \mu_0}$. This, however, is obviously in contradiction with the simple rule for how velocity should be transformed from one inertial frame to another moving relative to it.

Einstein's inner conviction, however, dictated that the reasoning based on the equivalence of two inertial frames was so natural that it must somehow be true. But if this is indeed the case, the following conclusion is unavoidable: light must propagate *with the same* speed *c* in all directions relative to every inertial frame, contradicting the well-known velocity transformation procedure referred to earlier.

The moment of enlightenment for Einstein was most likely when he realized that the difference between coordinate time and proper time (or—in other words—the "unrecognized possibilities" in the synchronization of distant clocks) makes it possible to resolve this contradiction.

First of all, we must clearly understand what it means that light propagates with the same speed *c* in all directions in every inertial frame.

Let us imagine inertial frames moving relative to each other as closed railway cars traveling at different speeds. Using Foucault's rotating-mirror method or Fizeau's rotating-disc method, in principle, the speed of light can be determined in all directions in each car, and in every case the same value *c* is obtained.

Since these are measurements of speed, time must somehow also be measured in such experiments, so there will certainly be clocks (assumed ideal) inside the railway cars. But in neither Foucault's nor Fizeau's experiment are clocks actually present. Only a *single clock* was used when calibrating the angular velocity of the motor rotating the mirror or the toothed disc. Thus, the time measured in these experiments is clearly proper time. The synchronization characteristic of coordinate time cannot come into play in these experiments.

In the Foucault and Fizeau experiments, it is necessary to determine the angular velocity of the disc or mirror, which can be done with any desired accuracy by measuring for a sufficiently long time using a single clock[4]. Determining the precise time of an instantaneous event is far more difficult. That is why, in the 19th century, light speed could not be measured "directly" using the formula *speed = distance/time*.

However, in a thought experiment, using this formula it is much easier to measure the speed of light between any two points P_1 and P_2, say in the $P_1 \to P_2$ direction, using *Einstein's synchronization procedure*.

Place a clock, $\mathcal{C}_1$, at P_1 and a clock, $\mathcal{C}_2$, at P_2. At time t_1, flash a lamp at P_1, the signal of which is reflected back toward P_1 by a mirror at P_2. Let the reflection occur at t_2, and let the moment the reflected signal returns to the origin be t_1'. According to our assumption, light travels with the same speed in the $P_1 \to P_2$ direction as in the opposite $P_2 \to P_1$ direction. Therefore, $\mathcal{C}_2$ is correctly synchronized with $\mathcal{C}_1$ when $t_2 = (t_1 + t_1')/2$. If it shows a smaller or larger value, it must be adjusted forward or backward accordingly.

Knowing the distance l between the two points, we can then send additional light signals and determine the speed of light between the two points using the formula $l/(t_2 - t_1)$ where, of course, t_1 and t_2 are the times of the light signal's departure from P_1 and arrival at P_2, measured with the already synchronized clocks[5]. If, in every inertial frame, the same value *c* is obtained for any pair of points, Einstein's hypothesis is confirmed: the speed of light is indeed the same in all directions in every inertial frame. If, on the other hand, the measured speeds vary, the hypothesis must be rejected.

Observe carefully: By the synchronization of two distant clocks, we no longer mean the Newtonian procedure, in which the positions of the clock hands are set to be the same at a common location. The sole criterion of synchronization is that the speed of light calculated using the formula $l/(t_2 - t_1)$ be equal to the constant *c*.

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At first glance, the constancy of the speed of light in railway cars moving at different velocities does not even seem particularly surprising. After all, if instead of the speed of light we were to determine the speed of bullets fired from identical types of pistols at rest in the cars, we would obtain the same result: the bullet would move with the same speed in every railway car (inertial frame) in all directions.

However, there is an enormous difference between light and a bullet! The speed of a bullet depends on the speed of the pistol, because it is constant relative to the pistol. This is why the speed of a bullet fired from a stationary gun is the same in every railway car.

According to Maxwell's equations, by contrast, the speed of light is independent of the speed of the light source (this is also supported by laboratory and astronomical observations). This fact is consistent with the prediction of the same equations that light is a wave phenomenon. Waves propagate with the same speed not relative to the source that generates them, but relative to the

[4] The relative error of the rotational speed is inversely proportional to the measurement time: $\Delta n/n = \Delta t/t$.
[5] In fact, on the round trip, the speed of light could already be determined using the formula $c = 2l/(t_1' - t_1)$.

medium that is oscillating[6]. Although in the case of light waves nothing points to the real existence of such a medium, before Einstein everyone believed that wave propagation was inconceivable without it. This hypothetical medium was the ether.

Thus, the assumption of the constancy of the speed of light in each of the railway cars realizing inertial frames resembles the behavior of a bullet (a particle), whereas its independence from the speed of the light source is characteristic of waves[7]. These two properties are, from an intuitive point of view, not at all compatible with one another.

Of course, these observations illustrated with railway cars cannot actually be carried out, because it is technically impossible to create inertial frames moving sufficiently fast relative to one another. Therefore, instead of encouraging such experiments, Einstein had to state his assumption in the form of a postulate:

The First Postulate of the Theory of Relativity:

If we were to measure the speed of light in every inertial frame in all directions, we would find that it has the same value and does not depend on the speed of the light source.

3. The ”correct synchronization” in relativity theory

If the speed of light is *indeed equal* to *c* in all directions in every inertial frame, then *it is possible and convenient* to synchronize the coordinate-time clocks $\mathcal{C}(x, y, z)$ in every inertial frame so that for a light signal propagating between any points P_1 and P_2 the following equality holds:

$$\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}{t_2 - t_1} = c. \qquad (4)$$

Here, t_1 is the moment the light signal is emitted as read on the coordinate-time clock at P_1, and t_2 is the moment it is received as read on the clock at P_2.

To achieve this, it is only necessary to individually synchronize the “scattered” $\mathcal{C}(x, y, z)$ clocks in the inertial frame with the clock at the origin $\mathcal{C}_0 \equiv \mathcal{C}(0,0,0)$ using Einstein’s procedure introduced in the previous chapter. Once this is done, equation (4) will hold for every point pair where one of the points is the origin.

However , it will also hold *for any arbitrary pair of points*, because *due to the constancy of the speed of light*, any pair of clocks $\mathcal{C}_1 \equiv \mathcal{C}(x_1, y_1, z_1)$ and $\mathcal{C}_2 \equiv \mathcal{C}(x_2, y_2, z_2)$ at rest in the inertial frame will

[6] As an example, consider the sound explosion, which occurs when an aircraft exceeds the speed of sound, and is caused by the piling up of sound waves whose speed does not increase with the speed of the aircraft.

[7] It is possible that this postulate already contains, in embryonic form, what we now briefly call the dual (particle and wave) nature of light. Incidentally, Einstein’s paper postulating light quanta preceded the theory of relativity by half a year.

also be synchronized with each other. (If the speed of light were not truly constant, no synchronization could enforce this.)

We can see this as follows: Place mirrors at P_0 and P_1 so that a light signal emitted from P_2 along the closed path $P_2 \to P_0 \to P_1 \to P_2$ returns to P_2. As a consequence of the first postulate, we have

$$l_{20} + l_{01} + l_{12} = c(t_2' - t_2),$$

where t_2 and t_2' are the readings of $\mathcal{C}_2$ at the moments of emission and reception of the light signal (their difference being *proper time*, is independent of coordinate time).

This equation can be rewritten as

$$l_{20} + l_{01} + l_{12} = c[(t_0 - t_2) + (t_1 - t_0) + (t_2' - t_1)].$$

On the right-hand side, t_0 and t_1 are the readings of $\mathcal{C}_0$ and $\mathcal{C}_1$ at the instants when the light signal is reflected by the mirrors at points P_0 and P_1.

Due to the synchronization with $\mathcal{C}_0$ we have $l_{20} = c(t_0 - t_2)$ and $l_{01} = c(t_1 - t_0)$. Therefore, it follows that $l_{12} = c(t_2' - t_1)$ which demonstrates that the clock pair $\mathcal{C}_1$ and $\mathcal{C}_2$ is also synchronized with each other.

Following general practice, from now on by *t* we shall always mean the coordinate time interpreted with Einstein's synchronization.

Let us now examine the properties of virtual ensembles of clocks that display coordinate time.

In Newtonian physics just as in relativity theory, inertial frames moving relative to one another carry along with themselves densely distributed virtual clocks that show the coordinate time.

According to the Newtonian conception of time, those two virtual clocks in *K'* and in *K* that are at a given moment located exactly above one another always show the same time[8], because, as we have already quoted, according to the *Principia* "*absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external*." With Einstein synchronization this is not the case: among the virtual clocks that "travel" together with *K*, *K'*, *K''*, etc., those that are momentarily at the same place ("overlap") all show different times.

Later we shall describe this difference in somewhat more detail, but I believe that it is in fact impossible to keep track mentally, all at once, of how the times read off from the set of overlapping virtual clocks at rest in *K* and traveling with *K'* are related to one another. But this is not necessary, because if we are familiar with the concepts of coordinate time and proper time, simple formulas are generally available to answer questions related to them. Let us look at one right away!

[8] However, the statement that, for example, the virtual clocks within *K* all show the same time *simultaneously* is meaningful only on the basis of the Newtonian conception of absolute time. In relativity theory, however, it is a tautology, because the qualifier "simultaneously" refers precisely to the simultaneity defined by the virtual clocks themselves.

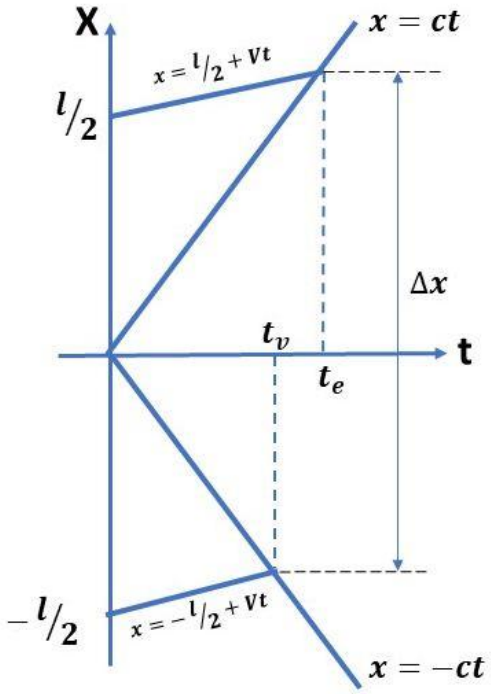

Figure 3

Problem: From the midpoint of a railway car of length *l* moving with speed *V*, we send out a light signal toward both ends of the car, where it triggers an explosion at each end. How much time elapses between the two explosions in the train's reference frame *K'* and in the track's reference frame *K*?

In *K'* the explosions are obviously simultaneous ($\Delta t' = 0$), because the speed of light in *K'* (as well) is the same c in both directions, and the light flash occurs at the center of the train.

If, however, we observe the events from *K*, we see the following (Fig. 3). The front of the train runs away from the forward-moving light signal with speed *V*, while the rear of the train approaches the backward-moving light signal with speed *V*. Since the speed of light is independent of the speed of the light source, the light signal also has speed *c* in *K* in both directions. Therefore, the distance between the front of the train and the light signal decreases with speed *(c − V)*, while the distance between the rear of the train and the light signal decreases with sped *(c + V)*.

If the length of the moving train is *l*, then on this basis the light signal reaches the front and the rear of the train after the coordinate times

$$t_e = \frac{l/2}{c - V} \quad \text{és} \quad t_v = \frac{l/2}{c + V} \tag{5}$$

respectively. From this it follows that, in the rest frame of the track, between the two explosions

$$\Delta t = t_e - t_v = \frac{l \cdot {}^{V}/_{c^2}}{1 - {}^{V^2}/_{c^2}} \tag{6}$$

seconds elapse, with the explosion at the front of the train occurring later. This Δt is obviously a difference in coordinate time, since it is a property of the worldlines plotted in the *t, x* coordinate plane. But how large a proper time difference $\Delta\tau$ elapses between the two explosions?

However, this question is not properly posed in this form, because the proper time between two events has meaning only with respect to *a single clock* that is located in the immediate vicinity of both events. Or, stated more simply: only a pair of Instantaneous events occurring on a given body has a well-defined proper time associated with it.

In the rest frame *K'* of the railway car, however, the two explosions are simultaneous, so a clock on which one could read off the proper time elapsed between them would have to move with infinite speed. Such a clock obviously does not exist.

From the point of view of *K*, on the other hand, the explosions do not occur at the same instant, so one could in principle imagine a clock that moves with exactly such a speed that it is present at the location of both explosions. But since *K* and *K'* are two equivalent inertial frames, such a clock cannot exist either.

In order to see more clearly what is at issue, let us calculate with what speed *U* the aforementioned clock would have to move relative to *K*.

If, in *K*, the two explosions occur at a spatial separation Δx, then clearly $U = \Delta x/\Delta t$. We have already found Δt in (6), and Δx can be calculated in a completely analogous way (see Fig. 3):

$$\Delta x = c \cdot (t_e + t_v) = c \cdot \left(\frac{l/2}{c - V} + \frac{l/2}{c + V} \right) = \frac{l}{1 - V^2/c^2}. \tag{7}$$

From (6) and (7) it follows that

$$U = \frac{\Delta x}{\Delta t} = \frac{c^2}{V}. \tag{8}$$

Now, we have tacitly assumed that the speed *V* of the train is less than the speed of light; otherwise the light signal traveling in the direction of motion would never reach the front of the train. As a consequence, *U* > *c*. In the rest frame of the train (*V* = 0) $U = \infty$,, and it is obvious that such a clock does not exist. But then, by virtue of the equivalence of inertial frames, no clock can exist that moves faster than light either.This line of reasoning quite clearly compels us to take the following restriction into account in the first postulate[9]:

1' Postulate:

(addition to the First Postulate):
No body can move, and no influence can propagate, faster than light.

[9] The necessity to inlude influences of any kind can be illustrated by the analysis of the train-tunnel paradox (see Appendix) which reveals that if, for example, elastic waves exceeded light velocity than no relativity of simultaneity could be maintained.

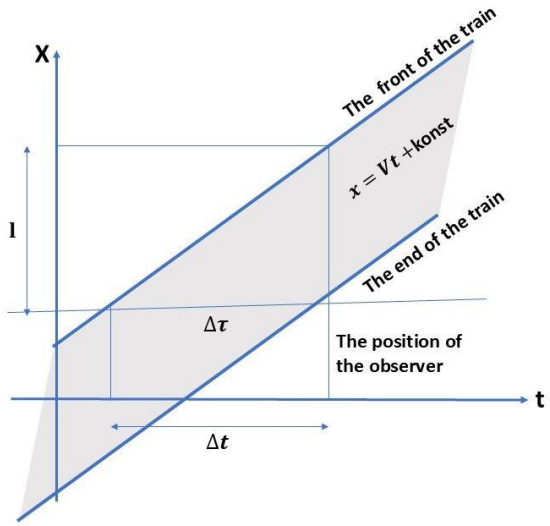

Figure 4

The set of questions we have just analyzed is briefly called the *relativity of simultaneity*. We can summarize it as follows: The simultaneity of two distant events cannot be interpreted as their occupying the same position in absolute time, because the concept of absolute time is empty. However, it is possible and useful to interpret simultaneity as the equality of coordinate time based on the constancy of the speed of light. Simultaneity defined in this way is, however, relative, because if two events are simultaneous in one inertial frame ((Δt=0), then in inertial frames moving relative to it they will no longer be simultaneous ($\Delta t \neq 0$): there will be inertial frames in which $\Delta t > 0$, and others in which $\Delta t < 0$. But this would cause a problem only if it were possible to determine the true temporal order of the two events using a moving clock. However, such a clock would have to move faster than light, which is not possible according to the (extended) first postulate. Event pairs of this type, whose temporal order is indeterminate, are called *spacelike*.

Let us return to equation (6) and show that it is, in principle, an experimentally verifiable relation. The measurement of the speed of light has already been discussed, so this is known. In order to measure Δt on the left-hand side, we must place real clocks showing coordinate time densely throughout space, select one of them, and synchronize the others to it using light signals. After this, t_e, t_v, and *V* can all be measured. But how can we measure the length l of a railway car that is in motion? The procedure is very simple: We must measure the time interval $\Delta\tau$ during which it passes a stationary observer, and multiply this proper time by the speed *V* of the train. This is the distance by which the front of the train is separated from the selected clock *at the moment t* when the rear passes the clock (see Fig. 4):

$$l = V \cdot \Delta\tau = V \cdot \Delta t. \tag{9}$$

***

The relativity of simultaneity has a catastrophic consequence: it pulls the ground out from under celestial mechanics, the most successful branch of Newtonian physics.

One of the Newton equations determining the motion of a binary star is

$$\ddot{\boldsymbol{r}}_1(t) = \frac{Gm_2}{|\boldsymbol{r}_2(t) - \boldsymbol{r}_1(t)|^3}\big(\boldsymbol{r}_2(t) - \boldsymbol{r}_1(t)\big).$$

According to this equation, the force acting on star 1 is determined by the position of star 2 *at the same instant* (action at a distance). As a consequence of the relativity of simultaneity, this equation certainly cannot be maintained in relativity theory.

The way toward a solution is shown by electrodynamics, where this problem does not arise, because the force acting on a point charge at time *t* is determined by the values at time *t* of the electric and magnetic fields at the location of the charge (*locality*), and the simultaneity of two events occurring *at the same place* is invariant[10].

With general relativity, Einstein solved the problem of gravitational interaction in a similar manner, because in this theory gravity, like electrodynamics, obeys locality.

4. Second postulate of relativity

Let us review the situation. What do we know? According to experience (Postulate 1), light propagates in every inertial frame, in every direction, with the same speed *c*. This fact makes possible and useful the Einsteinian choice of coordinate time (synchronization convention), according to which the equation of a light ray is, in every inertial frame, *x=ct*, or more generally

$$x = l \cdot c \cdot t + x_0 \qquad y = m \cdot c \cdot t + y_0 \qquad z = n \cdot c \cdot t + z_0 \qquad (l^2 + m^2 + n^2 = 1)$$

where *l,m,n* are the direction cosines of the light ray.

This removes the obstacle to the Maxwell equations having the same form in every inertial frame, since we do not run into a contradiction with the consequence of these equations that light propagates with speed *c* in all directions. The Maxwell equation written as an example in (1) can therefore be supplemented as follows:

[

$$K\colon \frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} + rot\boldsymbol{E}(\boldsymbol{r},t) = 0 \leftrightarrow K'\colon \frac{\partial \boldsymbol{B}'(\boldsymbol{r}',t')}{\partial t'} + rot\boldsymbol{E}'(\boldsymbol{r}',t') = 0. \qquad (10)$$

This type of connection is expressed in general form by the second postulate:

The Second Postulate of the Theory of Relativity:

Laws of physics are of the same form in every inertial frame

[10] The two events are the being there of the point charge and the field.

In Einstein's thought experiment on induction, it thus becomes meaningful that both methods yield the same electromotive force in the conductor, since the same Maxwell equations can be applied either in the rest frame of the magnet or in that of the coil (because both are inertial frames), and the deflection of the ammeter is invariant. This thought experiment also highlights that the field quantities ***E***, ***B*** change when we pass from one inertial frame to another (in the rest frame of the magnet $\boldsymbol{E} = 0$, whereas in that of the coil $\boldsymbol{E} \neq 0$). The exact transformation rules of the field quantities were found by Einstein already in his first paper[11]. The transformation $\boldsymbol{r}, t \rightarrow \boldsymbol{r}', t'$ is carried out by means of the Lorentz transformation, which will be discussed in Chapter 14.

Important remark on the relationship between the two postulates:

Question**:** From the second postulate it follows that the Maxwell equations are valid in exactly the same form in every inertial frame, and therefore that the speed of light is equal to *c* in every inertial frame and in every direction. Why, then, is the first postulate necessary, since it states the same thing?

Answer**:** The first postulate states an empirical fact that is independent of whether we are able to interpret it theoretically. Einstein—unlike his distinguished contemporaries H. Lorentz and M. Abraham—did not wish to *explain* the empirical constancy of the speed of light, but rather regarded it as *a guiding principle* that reveals the difference between proper time and coordinate time. This distinction made it possible for him, through relativity theory, to give an interpretation of Maxwell's electrodynamics that is consistent both with the constancy of the speed of light and with the equivalence of inertial frames, and that is capable of making a large number of—often surprising—predictions. The second postulate is therefore a theoretical generalization based on the empirically motivated first postulate, and it necessarily contains the statement of the first postulate as well.

5. The relationship between proper time and coordinate time

We will determine this relationship using the optical Doppler effect. It will be convenient to imagine that the transmitter emits sharp light pulses toward the receiver at regular time intervals, while the receiver moves with constant velocity. The Doppler effect consists in the fact that the time elapsed between two successive pulses is different at the receiver than at the transmitter $\Delta t_r \neq \Delta t_t$ (greater if they are moving away from each other, smaller if they are moving toward each other).

We call the *Doppler ratio (DR)* the ratio of the time interval between two successive signals as observed by the receiver to that emitted by the transmitter:

$$\boldsymbol{DR} = \frac{\boldsymbol{The\ time\ interval\ between\ the\ signals\ observed\ by\ the\ receiver}}{\boldsymbol{The\ time\ interval\ between\ the\ signals\ emitted\ by\ the\ transmitter}}$$

Let us calculate DR in the rest frame of the transmitter, in which the velocity of the receiver is $v < c$ (Fig. 5).

If the receiver were at rest, then Δt_r would be equal to Δt_t. But between two signals it moves a distance $v \cdot \Delta t_r$ farther away from the transmitter, therefore

[11] Earlier these transformations were known only approximately-

$$\Delta t_r = \Delta t_t + \frac{v}{c}\Delta t_r, \tag{11}$$

or

$$\frac{\Delta t_r}{\Delta t_t} = \frac{1}{1 - {}^{v}/_{c}}. \tag{12}$$

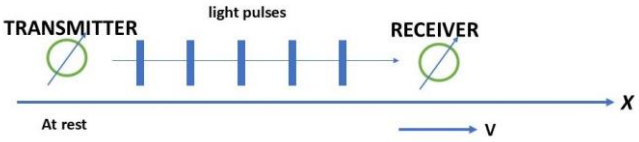

Figure 5

The Δt_t emitted by the transmitter and the Δt_r observed by the receiver are therefore *differences in coordinate time* between two successive signals, because equation (11) refers to the horizontal distance between the corresponding intersection points on the graph depicting the world lines of the transmitter, the receiver, and the light signals (Fig. 6):

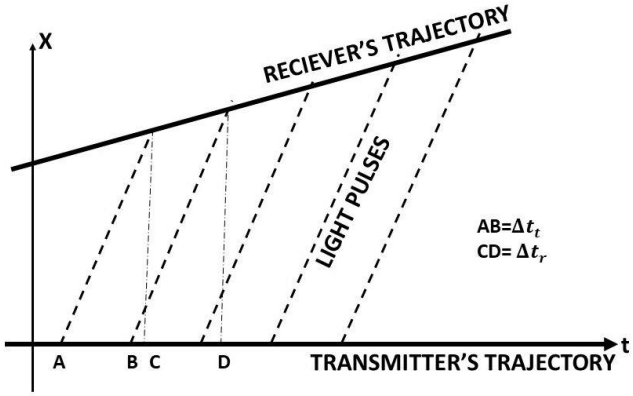

Figure 6

However, in the numerator and denominator of the Doppler ratio *DR* it is not these coordinate-time differences that appear, but the corresponding proper-time differences, as measured by the clocks associated with the emitter and the receiver:

$$DR = \frac{\Delta\tau_r}{\Delta\tau_t}. \tag{13}$$

To calculate the Doppler ratio, we therefore need to know the relationship between coordinate time and proper time, that is, the function F appearing in the formula $\Delta\tau = F(\Delta t, v, c)$.

The function F must have the dimension of seconds, which is only possible if F is proportional to Δt, with the proportionality factor being a function of the ratio ${}^{v^2}/_{c^2}$:

$$\Delta\tau = F(\Delta t, v, c) = \Phi\left(\frac{v^2}{c^2}\right) \cdot \Delta t. \tag{14}$$

For a clock at rest, $\Delta\tau = \Delta t$, therefore $\Phi(0) = 1$. Since the relationship between the two kinds of time cannot depend on the direction of motion, the function Φ can depend only on the square of the velocity.

Let us substitute this expression into equation (13):

$$DR = \frac{\Delta\tau_r}{\Delta\tau_t} = \frac{\Phi\left(\frac{v^2}{c^2}\right) \cdot \Delta t_v}{\Phi(0) \cdot \Delta t_a} = \Phi\left(\frac{v^2}{c^2}\right) \cdot \frac{\Delta t_v}{\Delta t_a} = \Phi\left(\frac{v^2}{c^2}\right) \cdot \frac{1}{1 - {}^{v}/_{c}}. \tag{15}$$

The single-variable function Φ is not yet known, but it can be determined if we also write the expression for *DR* in the rest frame of the receiver and equate it with that in (15).

Since the speed of light is of the same value *c* in the rest frame of the receiver as in that of the emitter, in the receiver's rest frame we have

$$\Delta t_r = \Delta t_t + \frac{v}{c}\Delta t_t, \tag{16}$$

and from this

$$\frac{\Delta t_r}{\Delta t_i} = 1 + {}^{v}/_{c}\,. \tag{17}$$

This relation can also be obtained from the diagram below; therefore, it also refers to coordinate time.

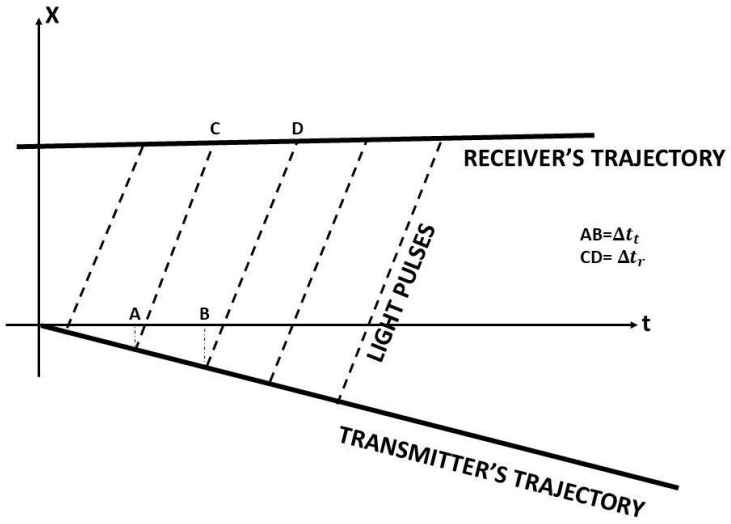

Figure 7

So then

$$DR = \frac{\Delta\tau_r}{\Delta\tau_t} = \frac{\Phi(0) \cdot \Delta t_r}{\Phi\left(\frac{v^2}{c^2}\right) \cdot \Delta t_t} = \frac{1}{\Phi\left(\frac{v^2}{c^2}\right)} \cdot \frac{\Delta t_r}{\Delta t_t} = \frac{1}{\Phi\left(\frac{v^2}{c^2}\right)} \cdot (1 + {}^{v}/_{c}). \tag{18}$$

From equating equations (15) and (18) we obtain the function

$$\Phi = \sqrt{1 - {v^2}/{c^2}} \tag{19}$$

with the help of which *DR* can be calculated either from (15) or from (18):

$$DR = \sqrt{\frac{1 + {v}/{c}}{1 - {v}/{c}}}. \tag{20}$$

But what we are really interested in now is not *DR*, but the relationship between coordinate time and proper time. Based on equations (14) and (19), this relationship is the following:

$$\boxed{\Delta\tau = \Delta t \cdot \sqrt{1 - {v^2}/{c^2}}} \tag{21}$$

As we shall see, all the well-known kinematical consequences of relativity theory — including the Lorentz transformation — already follow from this single equation. In the derivation of equation (21), the effectiveness of the Doppler effect is explained by the fact that, apart from the conceptual distinction between coordinate time and proper time, it was only necessary to exploit that the speed of light is the same in the rest frame of both the transmitter and the receiver. This assumption implicitly introduces Einstein synchronization into the line of reasoning.

6. Transformation of Velocity

In Chapter 2, in connection with the constancy of the speed of light (the first postulate), we said that "*this is impossible, because it is in obvious contradiction with the fundamental property of velocity, namely that it changes when we pass from one reference frame to another reference frame moving relative to it*." But Einstein realized that "*the difference between coordinate time and proper time makes it possible to resolve this contradiction*." The Doppler effect provides an opportunity for a very simple verification of this insight.

According to the *concept of relative velocity*, by the receiver's relative velocity with respect to the emitter we mean the velocity of the receiver measured in the rest frame of the transmitter. This was denoted by v. Similarly, the relative velocity of the transmitter with respect to the receiver is equal to $-v$. The quantity *DR* is invariant: no matter from which reference frame we determine its value, we must obtain the same result. Starting from this requirement, we obtained equation (21).

If we calculate *DR* in a reference frame moving with an arbitrary velocity along the *x*-axis and equate the result with equation (20), then we can read off how, according to relativity theory, velocities must be transformed between reference frames moving collinearly with respect to each other.

Let us consider the (left-moving) reference frame *K*, in which the velocities of the transmitter and the receiver satisfy $0 < v_t < v_r < c$. Then, since the transmitter approaches the receiver with velocity v_t, while the receiver recedes from it with velocity v_r, the analogue of equations (11) and (16) is the following:

$$\Delta t_r = \Delta t_t - \frac{v_t}{c}\Delta t_t + \frac{v_r}{c}\Delta t_r, \tag{22}$$

from which[12]

$$\frac{\Delta t_r}{\Delta t_t} = \frac{1 - {v_t}/{c}}{1 - {v_r}/{c}}, \tag{23}$$

As a result we obtain

$$DA = \frac{\Delta\tau_r}{\Delta\tau_t} = \frac{\Phi\left(\frac{v_r^2}{c^2}\right)\cdot\Delta t_r}{\Phi\left(\frac{v_t^2}{c^2}\right)\cdot\Delta t_t} = \frac{\sqrt{1 - {v_r^2}/{c^2}}}{\sqrt{1 - {v_t^2}/{c^2}}}\cdot\frac{\Delta t_r}{\Delta t_t} = \frac{\sqrt{\left(1 + {v_r}/{c}\right)\left(1 - {v_t}/{c}\right)}}{\sqrt{\left(1 - {v_r}/{c}\right)\left(1 + {v_t}/{c}\right)}}. \tag{24}$$

If we carry out the multiplications under the square root and divide both the numerator and the denominator by $\left(1 - {v_r v_t}/{c^2}\right)$, we arrive exactly at formula (20), in which

$$v = \frac{v_r - v_t}{1 - \frac{v_r v_t}{c^2}}.$$

Since in (20) the quantity v is the velocity of the receiver relative to the transmitter, this formula is the general expression for relative velocity.

The *velocity-addition* formula is obtained by rearranging this:

$$v_r = \frac{v + v_t}{1 + \frac{v v_t}{c^2}}.$$

With this formula one can calculate the velocity of the receiver in frame *K* when the velocity of the transmitter in *K* and the relative velocity of the receiver with respect to the transmitter are known.

The above formulas are, of course, not tied to the Doppler effect, therefore it is expedient to write them also with more neutral notation. Let us assume that the inertial frame *K'* moves with velocity *V* relative to *K*, and consider a point mass whose velocities relative to *K* and *K'* are v and v', respectively ($v \to v', v_t \to V, v_r \to v$). Then

$$v' = \frac{v - V}{1 - \frac{vV}{c^2}}; \qquad v = \frac{v' + V}{1 + \frac{v'V}{c^2}}. \tag{25}$$

Why do these formulas differ from their Newtonian counterparts[13] $v' = v - V$ and $v = v' + V$, which are obtained from (25) in the limit $c \to \infty$? Because the coordinate time flows differently in the inertial frame *K*, which is at rest, and in the moving inertial frame *K'*, and velocity must always be

[12] In the rest frame of the receiver, $v_r = 0$ and $v_t = -v$; therefore, we recover equation (17).

[13] We note that if v_1 and v_2 are the velocities of two bodies with respect to a *given* reference frame *K*, then $v_1 - v_2$ is the *rate of change of the distance* between the two bodies in *K*, according to both Newtonian physics and relativity theory. Conceptually, this is very different from the relative velocity of one body with respect to the other, but in Newtonian physics the rate of change of the distance and the relative velocity are numerically equal to each other. In relativity theory, this is not the case.

understood with respect to the coordinate time of the reference frame relative to which it is defined.

Now let us assume that instead of a point mass, a light signal propagates in *K*: $v = c$. According to the first equation of (25), the velocity v' of the light signal relative to *K'* is then also equal to c, regardless of the velocity V with which *K'* moves relative to *K*. And, as we see, it is precisely the distinction between proper time and coordinate time that makes this possible.

7. Time dilation

By substituting (3) into (21), we find that for a mass point (clock) moving with constant velocity V, the proper time runs more slowly ("is stretched," "dilates") than the coordinate time. If, in particular, the trajectory of the mass point is $x = Vt$, then between the points $x_1 = Vt_1$ and the later $x_2 = Vt_2$, the proper time $\Delta\tau$ elapsed on the clock is smaller than $\Delta t = t_2 - t_1$. This phenomenon is *time dilation*.

First of all, let us examine what follows from this for virtual clocks that display coordinate time. Consider the coordinate system *K'*, which moves with constant velocity *V* relative to the rest system *K* along their common *X* -axis. Both systems are "densely filled" with virtual clocks that display the coordinate times *t'* and *t*, respectively. Let us single out a virtual clock *C'* in *K'* which, "at the moment of selection," shows exactly the same time as the *K*-clock with which it coincides.

We will find that during its motion, the time read off on *C'*, as a consequence of time dilation, gradually lags behind the time read off on the *K* -virtual clocks with which it is momentarily coincident. However, since *K* and *K'* are two equivalent inertial systems, we will observe the same situation if we compare the reading of a clock C at rest in *K* with the readings of those *K'*-virtual clocks which it is momentarily passing.

These paradoxical properties of the virtual clocks associated with coordinate systems moving relative to one another do not logically contradict each other at all; nevertheless, to follow them simultaneously in one's imagination surely lies beyond the intellectual capacity of the average person.

***

In Newtonian physics, the cause of the Doppler effect is the continuous change of the distance between the source and the receiver (*longitudinal Doppler effect*). In relativity theory, the effect of time dilation is added to this. As a consequence of time dilation, however, a Doppler effect also occurs when the receiver moves on a circular orbit (with constant speed v) around the source, so that the distance between them does not change at all (*transverse Doppler effect*). In this case $\Delta t_v = \Delta t_a$, and[14]

$$DA = \frac{\Delta\tau_v}{\Delta\tau_a} = \frac{\Phi\left(\frac{v^2}{c^2}\right)\cdot\Delta t_v}{\Phi(0)\cdot\Delta t_a} = \Phi\left(\frac{v^2}{c^2}\right) = \sqrt{1 - {v^2}/{c^2}}\,. \qquad (26)$$

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[14] This follows from equation (28), because in uniform circular motion $v(t) = v = constant$).

Time dilation also occurs for motions with variable velocity. In the infinitesimal coordinate time interval $(t, t+dt)$, the increase of the proper time of a body moving with velocity $v(t)$ is, according to (21),

$$d\tau = dt \cdot \sqrt{1 - {v^2(t)}/{c^2}} \tag{27}$$

The *clock paradox* (or *twin paradox*) is a consequence of this formula (Fig. 8).

Let us consider clocks $\mathcal{C}_a$ and $\mathcal{C}_b$, which before the instant t_1 and after the later instant t_2 are located next to each other at the point *X=0* in an inertial frame, and which are synchronized before time t_1. Between the two instants, $\mathcal{C}_a$ "sets out on a journey" along the trajectory $X = x(t)$, on which its velocity is $\dot{x}(t) = v(t)$. According to (27), during its journey

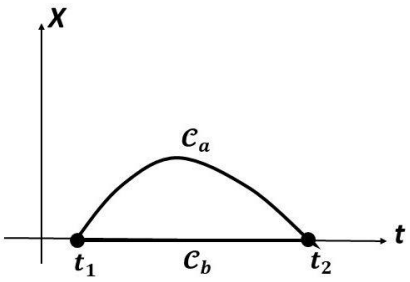

Figure 8

$$\Delta\tau_a = \int_{t_1}^{t_2} d\tau_a = \int_{t_1}^{t_2} dt \cdot \sqrt{1 - {v^2(t)}/{c^2}} \tag{28}$$

of proper time elapses on it. On $\mathcal{C}_b$, which remains at rest during the interval (t_1, t_2), obviously $\Delta\tau_b = t_2 - t_1$ of proper time elapses during the same interval. Since

$$\Delta\tau_a = \int_{t_1}^{t_2} d\tau_a = \int_{t_1}^{t_2} dt \cdot \sqrt{1 - {v^2(t)}/{c^2}} < \int_{t_1}^{t_2} dt = t_2 - t_1 = \Delta\tau_b, \tag{29}$$

after t_2, when they again run at the same rate, they will no longer be synchronized with each other; rather, $\mathcal{C}_a$ will lag behind $\mathcal{C}_b$. This phenomenon is the clock paradox.

General conclusion: If several clocks moving along different paths all encounter the same two given events, then they generally measure different proper-time intervals between the events. The proper-time interval is greatest on the clock that did not undergo acceleration between the two events.

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The Newtonian synchronization of (virtual) clocks that display coordinate time, discussed on page 4, cannot be realized even with ideal clocks because of the clock paradox: clocks that are synchronized at one location become desynchronized during their transport, or, if clocks that have already been placed are adjusted to a standard clock, the rate of the standard clock changes during the process of carrying it around. In Einstein synchronization, this danger does not arise, because the clocks are synchronized at their final place, using light signals.

8. Lorentz Contraction

In Chapter 3 we saw how the length of a moving train can be measured. An observer standing on the embankment must measure the time interval $\Delta\tau$ during which the train passes by him. Then, using the formula $l = V \cdot \Delta\tau$, he can easily calculate the length of the moving train.

Meanwhile, those sitting on the train[15] see that the observer standing on the track appears at the front of the train and moves with speed *V* relative to the train until he disappears at the rear of the train. If the *X'*-axis in the car is chosen such that its positive direction coincides with the direction of motion, then the trajectory of the observer is shown by the graph in Fig. 9.

In the graph, l_0 is the length of the railway car, which the passengers determine using a meter rod, while $\Delta t'$ is the coordinate time interval during which the observer passes alongside the car, and which the passengers can in principle read off from properly synchronized virtual clocks densely placed inside the car.

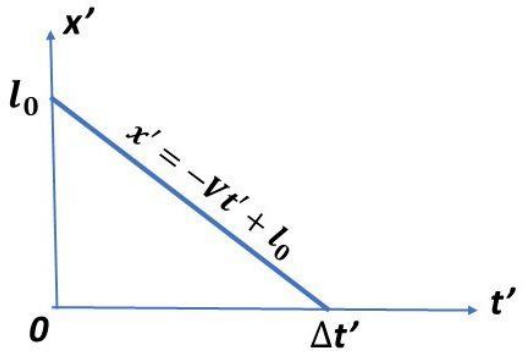

Figure 9

There is no reason to assume a priori that $l_0 = l$. In fact, they are not equal, because

$$l = V \cdot \Delta\tau = V \cdot \Delta t' \cdot \sqrt{1 - {V^2}/{c^2}} = l_0 \cdot \sqrt{1 - {V^2}/{c^2}} \qquad (30)$$

We have used Eq. (21) and the fact that, based on Fig. 9, $l_0 = V \cdot \Delta t'$.

According to Eq. (30), the length of the train is shorter when viewed from the inertial frame with respect to which it is moving than from the frame in which it is at rest. This phenomenon is the *Lorentz contraction*, which is made possible by the relativity of simultaneity: with the moment when the rear end of the train passes the observer measuring the train's length (see Fig. 4), different moments in time are simultaneous, in different inertial frames, with the position of the front of the train along its trajectory.

***

The dimensions perpendicular to the motion (the height and width of the train) do not change when referred to inertial frames moving at different velocities. Let us imagine a gate whose inner

[15] For the sake of simplicity, let us imagine that the train consists of a single long motor car.

dimensions almost exactly match the height and width of the train at rest, so that the stationary train just fits inside it. If the transverse dimensions of moving bodies were to change—say, if they were to contract—then, viewed from the rest frame of the gate, the moving train could easily pass through the gate, whereas from the rest frame of the train the gate would collide with the train; therefore, the two inertial frames would not be equivalent to each other. The equivalence of inertial frames thus requires that transverse dimensions do not change when referred to different inertial frames.

A nearly identical paradox, however, can also be formulated in connection with Lorentz contraction. Let us imagine a tunnel that is exactly as long as the proper length of a train. In this case, viewed from the rest frame of the tunnel, the moving train completely disappears into the tunnel for a while, whereas in the rest frame of the train at least one of its ends is always sticking out. Why do we not regard these, too, as mutually contradictory events, on the basis of which Lorentz contraction itself should be rejected?

"Transverse contraction," viewed from two inertial frames in uniform motion relative to each other, would initialize two completely different chains of events. From this it would follow that the two inertial frames are not equivalent to each other. If, therefore, we wish to preserve this equivalence, we must assume that Lorentz contraction has no analogue in the transverse direction.

The "train–tunnel" paradox associated with Lorentz contraction, however, is merely a special consequence of the relativity of simultaneity; it does not involve the initiation of two different chains of events when we observe the situation from the rest frame of the tunnel or from that of the train. Therefore, it is just as compatible with the equivalence of inertial frames as the relativity of simultaneity itself. If we nevertheless try to force two different chains of events to occur from the paradox when viewed from the two perspectives, then the reasoning leading to one of them must certainly be flawed, because it fails to take into account the relativity of simultaneity (see the Appendix).

9. An Illustrative Problem

A cyclist rides with speed v along a straight section of a country road. On a railway embankment parallel to the road, a train of proper length l_0 approaches him with speed V. How long does it take for the cyclist to pass alongside the train, according to his own wristwatch?

According to Newtonian physics, this takes a time $\frac{l_0}{V+v}$, since the distance ***s*** between the end of the train and the cyclist decreases with speed $V + v$.

For the relativistic solution, we first choose a coordinate system. Let this be *K*, fixed to the railway embankment.

How far is the rear end of the train from the cyclist at the moment when he just reaches the train? Due to the Lorentz contraction of the train, the distance is $l_0\sqrt{1-\frac{V^2}{c^2}}$.

How long does it take for this distance to decrease to zero?

$$\frac{l_0\sqrt{1-V^2/c^2}}{V+v}$$

seconds[16].

Is this a proper-time interval or a coordinate-time interval?

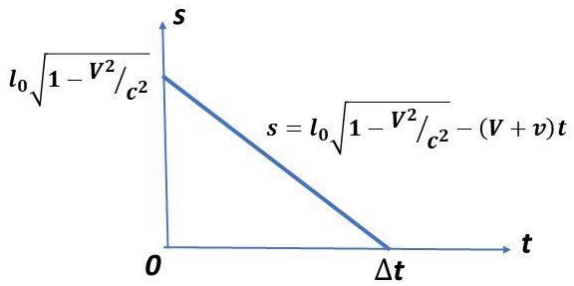

Figure 10

Fig. 10 shows that it is coordinate time:

$$\Delta t = \frac{l_0\sqrt{1-V^2/c^2}}{V+v}.$$

However, the cyclist's wristwatch shows his *proper time*, so the solution of the problem is

$$\Delta\tau = \Delta t \cdot \sqrt{1-v^2/c^2} = \frac{l_0\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{V^2}{c^2}}}{V+v}. \qquad (31)$$

Now let us solve the same problem in the rest frame *K'* of the train. Let us denote the cyclist's speed in *K'* by v'.

Obviously, in *K'* the cyclist pedals alongside the train for a coordinate time

$$\Delta t' = l_0/v'$$

During this interval, the proper time on his wristwatch is

$$\Delta\tau = \Delta t' \cdot \sqrt{1-v'^2/c^2} = l_0\frac{\sqrt{1-v'^2/c^2}}{v'} \qquad (32)$$

The time elapsed on the cyclist's wristwatch, however, cannot depend on whether we perform the calculation in the rest frame of the railway embankment or the train. That is why we did not put a prime on $\Delta\tau$ in (32). An important check of the correctness of our formulas is to verify that the right-hand sides of (31) and (32) can indeed be transformed into one another.

[16] In the denominator of this formula we do not use Eq. (25), because we are not required to calculate the velocity of some body as a *relative velocity*, but rather the *rate of change of the distance* between two bodies within a given coordinate system (see footnote 13).

The problem, of course, is to express v' in (32) in terms of v. In Newtonian physics, $v' = v + V$. In relativity, this changes[17]. According to (25):

$$v' = \frac{v + V}{1 + {vV}/{c^2}}. \tag{33}$$

Substituting (33) into the denominator of (32) and multiplying both numerator and denominator by $\left(1 + {vV}/{c^2}\right)$, we get

$$\Delta\tau = l_0 \frac{\sqrt{\left(1 + {vV}/{c^2}\right)^2 \left(1 - {v'^2}/{c^2}\right)}}{v + V}. \tag{34}$$

Since $\left(1 + {vV}/{c^2}\right)^2 v'^2 = (v + V)^2$, we obtain

$$\Delta\tau = l_0 \frac{\sqrt{\left(1 + {vV}/{c^2}\right)^2 - {(v + V)^2}/{c^2}}}{v + V} = l_0 \frac{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{1 - \frac{V^2}{c^2}}}{V + v},$$

which is indeed identical to (31).

This is by no means a trivial result. It shows that our formulas "know" that the time elapsed on the cyclist's watch while passing the train is *independent of the chosen frame of reference* (proper time is invariant). In general form, this will be proven in Chapter 14.

Coordinate-time intervals, however, are not equal: $\Delta t' \neq \Delta t$. But this is not a problem, because the two coordinate time intervals could, in principle, be read off on different sets of clocks—the (virtual) clocks fixed to *K* and to *K'*, respectively.

10. Acceleration

In relativity theory there are two kinds of acceleration: the *proper acceleration* α and the *coordinate acceleration* a, which can be put in parallel with the proper time τ and the coordinate time *t*.

Let a (pointlike) spaceship move along the *X*-axis in the inertial frame *K*. By the proper acceleration α of the spaceship we mean the acceleration indicated by the accelerometer on board the spaceship (and perceived by the astronaut). Since the reading of an accelerometer is a fact independent of the observer's point of view, just like the reading of a clock carried on the spaceship, the proper acceleration is invariant in the same way as the proper time.

By the coordinate acceleration of the spaceship we mean its acceleration in *K* relative to the *X* coordinate axis:

$$a = \frac{dv}{dt}. \tag{35}$$

When we simply say "acceleration," we almost always mean the coordinate acceleration.

[17] Let the cyclist's velocity be in the positive direction. Then the cyclist's velocity relative to the train (v') is equal to the cyclist's velocity relative to the embankment (v) plus the embankment's velocity relative to the train (V). Equation (33) expresses this relationship relativistically.

Now we must find the relation between the two kinds of acceleration, the analogue of (21).

The accelerometer of a spaceship moving with constant velocity reads zero. If, according to the astronaut's own clock, the engine is switched on with constant intensity for an infinitesimally short proper time $d\tau$, the accelerometer jumps to α, and after the engine is switched off it returns to zero. From this, the astronaut can determine that the velocity has increased by

$$dv_0 = \alpha \cdot d\tau \tag{36}$$

compared to the velocity before the acceleration, which was of course equal to zero in the astronaut's rest frame.

For an external observer in frame *K*, this increase in velocity takes place from v to $v + dv$, which, on the basis of (25), is given by

$$v + dv = \frac{v + dv_0}{1 + \frac{v dv_0}{c^2}}. \tag{37}$$

Our line of reasoning refers to infinitesimally small $d\tau$, dv_0, and dv; therefore, the right-hand side of (37) is valid only to linear order in dv_0. On the basis of the sum formula for the infinite geometric series with ratio $-v dv_0/c^2$

$$\frac{1}{1 + \frac{v dv_0}{c^2}} = 1 - \frac{v dv_0}{c^2} + \left(\frac{v dv_0}{c^2}\right)^2 - \cdots \cong 1 - \frac{v dv_0}{c^2} \qquad \left(\frac{v dv_0}{c^2} \ll 1\right).$$

If we substitute this into the right-hand side of (37), multiply the two binomials, and—apart from v—keep only terms linear in dv_0, we arrive at the equation

$$dv = \left(1 - {v^2}/{c^2}\right) dv_0 \tag{38}$$

However, according to (36) and (27) $dv_0 =$"α" $\cdot\, d\tau = \alpha \cdot \sqrt{1 - {v^2}/{c^2}} \cdot dt$. If we substitute this[18] into the right-hand side of (38) and make use of (35), we obtain the relation between the proper acceleration and the coordinate acceleration:

$$a = \left(1 - {v^2}/{c^2}\right)^{3/2} \alpha. \tag{39}$$

The coordinate acceleration is always smaller than the proper acceleration. This property is the *acceleration suppression*[19]. Clearly, a relationship of exactly this type between accelerations is

[18] This formula shows how time dilation contributes *directly* to the coordinate acceleration, in addition to the *indirect* contribution mediated by the velocity-addition law. This direct effect expresses the fact that, although the fuel consumption is constant according to the astronaut's clock, an observer at rest in *K* perceives it as a continuously decreasing rate of fuel consumption.

[19] The terms *coordinate acceleration*, *proper acceleration*, and *acceleration suppression* are my own suggestions and are not widely used. Accordingly, the lengths l_0 and l (Sect. 8) may be referred to as the *proper length* and the *coordinate length*, respectively.

required in order for our example rocket never to be able to reach the speed of light in *K* —even in the case of constant α. In the Newtonian limiting case $v^2/c^2 \ll 1$, we have $\alpha = a$; therefore, in classical mechanics it is not necessary to distinguish between the two kinds of acceleration.

The attentive reader will surely notice that our line of reasoning began with a very special assumption: we assumed that the spaceship accelerates only for an infinitesimally short time, and that its velocity is constant before and after. This picture, however, served only to make the argument easier to understand; in fact, in none of our steps did we make use of how the spaceship moves before and after the acceleration lasting for the time interval $d\tau$. Consequently, equation (39) is applicable to motion with arbitrarily varying velocity as well.

We have a very useful conceptual tool that makes it possible for our reasoning to be immediately applicable to an arbitrarily accelerated object. This is the concept of the *instantaneous rest frame*.

Let a point mass move in space with an arbitrarily varying velocity **v**(t). Choose an arbitrary instant t_0, and select the inertial coordinate system that moves with the *constant* velocity **v**(t_0); denote it by K_0. Since the body is at rest in K_0 at the instant $t=t_0$, we call this coordinate system the instantaneous rest frame (at t_0). K_0 is the reference frame in which the acceleration is equal to the proper acceleration α_0, and the increase in velocity relative to this frame is given by equation (36).

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Up to this point we have been dealing with questions of relativistic kinematics. One important kinematical task still remains: the derivation of the Lorentz transformation. We postpone this to a later stage and now turn to the presentation of the foundations of relativistic dynamics.

11. The relativistic equation of motion

Einstein already showed in his June 1905 foundational paper how to find the relativistic generalization of the Newtonian equation of motion $m\boldsymbol{a}=\boldsymbol{F}$. One must start from the fact that the Newtonian equation of motion is an excellent approximation at low velocities (when $v \ll c$). This requirement can be formulated precisely by demanding that, for a body at rest (that is, in the instantaneous rest frame K_0), it holds in unchanged form: $m_0\boldsymbol{a}_0 = \boldsymbol{F}_0$. After this, all that remains is to express the quantities with the zero index, which refer to the instantaneous rest frame, in terms of the corresponding quantities valid in the frame *K* in which the body is moving with the instantaneous velocity $\boldsymbol{v}$.

The appropriate rewriting of the right-hand side depends on what kind of force is involved. In physical practice, only gravity and the electromagnetic force can give rise to motions at such high velocities that a relativistic treatment is required. According to general relativity, however, gravity is in fact not a force, and its effect is not correctly described by the Newtonian equation. For a point charge at rest in an electromagnetic field, only the Coulomb force acts; therefore, in practice, on the right-hand side of the equation $m_0\boldsymbol{a}_0 = \boldsymbol{F}_0$ we always have $\boldsymbol{F}_0 = Q\boldsymbol{\mathcal{E}_0}$ (the instantaneous rest frame may contain a magnetic induction $\boldsymbol{B_0}$, but this does not exert a force on a charge that is at rest).

On the left-hand side, the acceleration $\boldsymbol{a_0} \equiv \boldsymbol{\alpha}$ must be expressed in terms of the coordinate acceleration $\boldsymbol{a}$. The formula used for this purpose, equation (39), however, is valid only for

rectilinear motion, so in what follows we restrict ourselves to this case. Then, since a_0 is identical to the proper acceleration α, according to (39) we have

$$a_0 = \frac{a}{\left(1 - {v^2}/{c^2}\right)^{3/2}}. \tag{40}$$

If we substitute this expression into the one-dimensional equation of motion valid in *K₀*, $m_0 a_0 = F_0$ we obtain

$$m_0 \frac{a}{\left(1 - {v^2}/{c^2}\right)^{3/2}} = F_0 \tag{41}$$

where $F_0 = Q\mathcal{E}_0$.

Now we must decide whether there is any difference between the mass m_0 of a body at rest and the mass m of a moving body. Einstein postulated that in relativity theory the mass is a constant parameter of the body, independent of its motion, just as in Newtonian physics[20], that is,

$$m_0 = m. \tag{42}$$

If we make use of this in equation (41), then instead of (41) we may write

$$m \frac{a}{\left(1 - {v^2}/{c^2}\right)^{3/2}} = F_0.$$

On the right-hand side, however, the force $F_0 = Q\mathcal{E}_0$ defined in the frame *K₀* still appears, and this must also be expressed in terms of quantities measured in the frame *K*. We assume that the charge Q has the same value in every reference frame, but what about $\mathcal{E}_0$?

As already pointed out in Chapter 4 that, when transforming between different inertial coordinate systems, the electromagnetic field does not remain unchanged, and in frame *K* there generally appear components of the Coulomb and Lorentz forces that deflect the charge from the assumed straight-line direction. *In the special case, however, when a point charge moves in a homogeneous electric field parallel to the field strength*, the electric field in *K* remains the same as it was in *K₀*: $\mathcal{E}_0 = \mathcal{E} = constant$. As a consequence, the charge accelerates or decelerates, but its motion remains rectilinear. In this case,

$$ma = \left(1 - {v^2}/{c^2}\right)^{3/2} F, \tag{43}$$

[20] According to the widespread but incorrect conception, the mass increases with the speed of the body according to the formula

$$m = \frac{m_0}{\sqrt{1 - {v^2}/{c^2}}} \qquad (not\ true)$$

This nonexistent phenomenon is called “relativistic mass increase.” If this formula were valid instead of (42), then the exponent 3/2 would be missing from the equation of motion (it would be 1 instead). Experience—and, moreover, the entire structure of relativity theory—however, confirms equation (43).

where $F = Q\mathcal{E}$. On the right-hand side of the relativistic equation of motion (43), the characteristic factor $\left(1 - {v^2}/{c^2}\right)^{3/2}$ is a direct consequence of the acceleration suppression. It implies that *a given force is less and less able to accelerate a body as its velocity approaches the speed of light*, and as a result the velocity can approach the speed of light arbitrarily closely but can never reach it[21]. However, an accelerometer rigidly attached to the body would continuously indicate the same proper acceleration, $\alpha = Q\mathcal{E}/m$.

The relativistic formula for momentum must be derived, on the basis of the general definition of momentum, from the equation of motion. Momentum *p* is the quantity that is changed by an impulse[22]: $dp = F \cdot dt$. In Newtonian physics, $F \cdot dt = ma \cdot dt = d(mv)$, from which we obtain the momentum formula $p = mv$. In relativity theory, as a consequence of (43), this line of reasoning is modified as follows:

$$F \cdot dt = \frac{ma \cdot dt}{\left(1 - {v^2}/{c^2}\right)^{3/2}} = \frac{m \cdot dv}{\left(1 - {v^2}/{c^2}\right)^{3/2}} = d\left(\frac{mv}{\sqrt{1 - {v^2}/{c^2}}}\right). \tag{44}$$

From this it can be seen that the formula for momentum in relativity theory is the following:

$$p = \frac{mv}{\sqrt{1 - {v^2}/{c^2}}} = m\frac{dx/dt}{d\tau/dt} = m\frac{dx}{d\tau}. \tag{45}$$

It is also worth reflecting on what should properly be meant by the inertia of a body. The quantity ${F}/{a} = {QE}/{a}$ appears to be the most suitable, since inertia is greater the larger the force required to produce a given acceleration (or, equivalently, the smaller the acceleration produced by a given force). On the basis of (43), inertia would then be equal to the ratio $m/(1 - v^2/c^2)^{3/2}$.However, in this expression the denominator, which is a consequence of the acceleration deficit, is not a property of the body itself but depends on the choice of coordinate system. Yet we conceive the inertia of a body as an internal (invariant) property. Therefore, it is appropriate to understand the measure of inertia as the ratio ${F_0}/{a_0}$ referring to the body at rest, that is, simply the mass *m* of the body in relativity theory just as in Newtonian dynamics.

12. Kinetic energy

A body moving with velocity *V* has a *kinetic energy K* equal to the work done by the force acting on the body while it accelerates the body from rest to the velocity *V*. In Newtonian physics, this definition leads to the well-known formula $K = \frac{1}{2}mV^2$:

[21] It is completely incorrect to claim, as is often heard, that reaching the speed of light is prevented by the above-mentioned "relativistic mass increase," since this obviously contradicts the fundamental equality (42).

[22] Newton already interpreted the relationship between force and momentum in this way in the *Principia*.

$$K = \int_0^S Fds = \int_0^T Fv \cdot dt = m\int_0^T a \cdot v \cdot dt = m\int_0^T \frac{dv}{dt} \cdot v \cdot dt = m\int_0^V dv \cdot v = \frac{1}{2}mV^2 \quad (46)$$

The same definition in relativity theory yields the formula

$$K = \int_0^S Fds = \int_0^T Fv \cdot dt = m\int_0^T \frac{a}{(1 - v^2/c^2)^{3/2}} \cdot v \cdot dt =$$

$$= m\int_0^V \frac{dv \cdot v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = mc^2 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\Big|_0^V = \frac{mc^2}{\sqrt{1 - \frac{V^2}{c^2}}} - mc^2 . \quad (47)$$

The origin of the difference between the Newtonian and relativistic formulas is the characteristic denominator $(1 - v^2/c^2)^{3/2}$, which we know to be a *consequence of the acceleration suppression*.

As $V \to c$, the kinetic energy in relativity theory tends to infinity. The acceleration suppression therefore increases the kinetic energy, because a given velocity can only be reached over a longer distance than when the acceleration suppression is zero, and over this longer distance the force performs more work. This is the reason why the kinetic energy corresponding to a given V is greater in relativity theory than in Newtonian physics.

As we see, in Newtonian physics the kinetic energy is given by a single-term formula, whereas in relativity theory it consists of two terms. However, this by no means implies that the two terms separately possess independent physical significance. It merely reflects the fact that the value of the indefinite integral at the lower limit is zero in Newtonian physics, while in relativity theory it differs from zero. In his June 1905 paper, Einstein makes no comment at all on this two-term structure. Three months later, however, through the analysis of a completely independent thought experiment, he shows[23] that the expression mc^2 is equal to the internal energy E_0 of the body.

13. The mass-energy relation

By the internal energy E_0 we mean the sum of the various types of energy (thermal, chemical, electromagnetic, nuclear, subnuclear, etc.) stored in a body at rest. For this reason, E_0 is also called the rest energy. However, on the basis of this definition, E_0 can neither be calculated nor measured, because for that we know far too little about the structure of matter. At most, we can make definite statements about changes in the internal energy ΔE_0 in cases where the type of energy involved (e.g. thermal energy) is sufficiently well understood.

This situation is fundamentally changed by the mass–energy relation

[23] In the derivation of equation (47), we tacitly assumed that the work done by the accelerating force does not change the internal energy, that is, like the mass, it is independent of the velocity of the body. However, from the formula for kinetic energy we can learn nothing about whether mass and internal energy might be related to each other.

$$E_0 = mc^2 \tag{48}$$

In September 1905, in his short paper *Does the Inertia of a Body Depend upon Its Energy Content?*, Einstein arrived at this formula by means of the following thought experiment.

Let a body with internal energy E_0 and mass m be at rest in the inertial frame *K.* At a given moment, the body emits two completely identical electromagnetic signals (wave packets) in exactly opposite directions. The total momentum carried away by the two signals is obviously zero; therefore, after the emission, the body remains at rest in *K*.

Its internal energy, however, decreases from E_0 to some $\bar{E}_0$. If, in *K*, the energy of each signal is equal to $\epsilon/2$, then by energy conservation in *K*

$$E_0 = \bar{E}_0 + \epsilon. \tag{49}$$

Now let us observe *the same process* from a frame *K*' that moves with constant velocity v relative to *K*. Clearly, in *K*' the body moves uniformly with velocity v in the opposite direction, but neither the energy of the body nor that of the wave packets will be the same in *K*' as in *K*.

In his June paper, Einstein had already calculated how the energy of a wave packet moving in an arbitrary direction changes when transforming from *K* to *K*'. Applying this formula to the two wave packets moving in opposite directions, he found that their total energy in *K*' is equal to $\epsilon/\sqrt{1 - v^2/c^2}$. In addition to its internal energy, the body also possesses kinetic energy according to equation (47).

The next step is to write down energy conservation in *K*'. After the first attempt it becomes clear[24] that energy can be conserved in *K*' only if the mass of the body changes during the emission. Denoting the changed mass by $\bar{m}$, energy conservation in *K*' takes the form

$$E_0 + K(v, m) = \bar{E}_0 + K(v, \bar{m}) + \frac{\epsilon}{\sqrt{1 - v^2/c^2}},$$

or

$$K(v, m) + \epsilon = K(v, \bar{m}) + \frac{\epsilon}{\sqrt{1 - v^2/c^2}}. \tag{50}$$

Here $K(v, m)$ and $K(v, \bar{m})$ are the kinetic energies of the body before and after the emission, whose formula Einstein had already derived in his June paper. This is equation (47), which we can substitute into (50):

$$\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 + \epsilon = \frac{\bar{m}c^2}{\sqrt{1 - v^2/c^2}} - \bar{m}c^2 + \frac{\epsilon}{\sqrt{1 - v^2/c^2}},$$

that is,

[24] If, on the right-hand side of (50), the kinetic energy contained the same mass *m* as on the left-hand side, then this term could be canceled, and we would arrive at a contradiction with (49). It is important to note that this change in mass does not violate the conservation of momentum in the emission process under consideration.

$$(m-\bar{m})\cdot c^2\cdot\left[\frac{1}{\sqrt{1-v^2/c^2}}-1\right]=\epsilon\cdot\left[\frac{1}{\sqrt{1-v^2/c^2}}-1\right].$$

The expression in brackets can obviously be canceled, and introducing the notations $\epsilon\equiv E_0-\bar{E}_0=\Delta E_0$ and $\Delta m=m-\bar{m}$, we obtain

$$\Delta E_0=\Delta m\cdot c^2 \tag{51}$$

which is the differential form of the mass–energy relation (48): the change in mass is proportional to the change in internal energy.

Einstein, however, was convinced that the formula is valid also in the original, non-differential form (48), that is, that there is no part of mass to which no energy corresponds. In the subsequent decades this conviction was fully confirmed by important experiments[25], but it became clear much earlier that the non-differential law (48) is the one that fits harmoniously into the structure of relativity theory.

Let E denote the *total* (kinetic plus internal) energy of a freely moving body. As a consequence of (48),

$$E\equiv K+E_0=\frac{mc^2}{\sqrt{1-\frac{V^2}{c^2}}}=mc^2\frac{dt}{d\tau}. \tag{52}$$

When particles transform into one another as a result of collisions or decays, this is the energy that is conserved. From (45) and (52) it follows that[26]

$$\left(E/c\,,p_x,p_y,p_z\right)=m\left(\frac{d(ct)}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right). \tag{53}$$

This quantity, called the four-momentum, which plays a fundamental role in relativity theory, is therefore a consequence of the non-differential mass–energy relation (48) (and not the other way around).

The non-differential mass–energy relation $E_0=mc^2$ finds its theoretical justification in general relativity. General relativity, the relativistic theory of gravitation, reached its final form in 1915. However, Einstein already proved in 1908 that if (48) is valid for inertial mass, which produces acceleration in the equation of motion, then it is also valid for gravitational mass, which can be determined with a spring balance.

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[25] The relation $E_0=mc^2$ is demonstrated, for example, by the annihilation of positronium at rest, $(e^-e^+)\rightarrow 2\gamma$ The electron mass m_e is well known, since its charge and its motion in an electromagnetic field are known; expressed in energy units, $m_e\approx$ 0.5 MeV. The positron is the antiparticle of the electron, and therefore its mass is equal to that of the electron. Accordingly, the mass of positronium is m≈1 MeV. The rest energy E_0 of positronium is equal to the energy of the two gamma photons, which can be measured accurately. The values of *m* and E_0 obtained in these two independent experiments satisfy the relation $E_0=mc^2$ (that is, the total energy of the two gamma photons is also approximately 1 MeV.

[26] We have placed the factors of *c* in such a way that all components of the four-component quantity have the same physical dimension.

Since in a decay or collision process it is only the total energy that is conserved, while the internal energies of the individual particles are not, according to relativity theory—and thus in physics in general—there is no conservation of mass.

However, in the class of phenomena that form the basis of Newtonian physics, the internal energy of bodies either does not change at all (for example, in celestial mechanics), or, if it does change, the corresponding change in mass is negligibly small compared to the masses involved. Therefore, in such cases mass is conserved with high accuracy (but only approximately!).

Abandoning mass conservation is unacceptable to many, perhaps because they identify the concept of mass with the concept of matter—although these are two different notions. They think that if mass is not conserved, then the principle that "matter is not destroyed, only transformed" is endangered.

The term *mass* is a technical one; it is equal to the quantity that multiplies the acceleration in Newton's equation. For example, no mass can be assigned to the electromagnetic field, because acceleration does not appear in Maxwell's equations. Nevertheless, the electromagnetic field is also regarded as a form of matter. In the electron–positron annihilation $(e^- e^+) \to 2\gamma$ for instance, massive matter is transformed into massless electromagnetic radiation. Both objects are of material nature, but only one of them possesses mass: matter is not destroyed, only transformed.

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Important note: The mass–energy relation (48) assigns a definite internal energy to every body that has mass, but it does *not* assign mass to every type of energy. Since the internal energy coincides with the energy of a body *at rest*, it cannot be applied, for example, to an electromagnetic wave packet or a photon, because these have no rest frame.

Similarly, (48) cannot be applied to the kinetic energy *K* or the total energy *E* of a moving body. That is, neither K/c^2 nor E/c^2 equals the mass of a moving body, which—according to (42)—is always simply m.

14. The Lorentz transformation

In the previous three chapters we presented the foundations of relativistic dynamics. Now we return to kinematics and, using our earlier results, derive the formulas of the Lorentz transformation.

Let us start from the following problem:

A train of proper length $l_0 \equiv \Delta x_0$ (its rest frame is K_0) moves with speed *V* along the track (frame *K*). From the rear of the train a bullet is fired toward the front of the train; it reaches the front in a time Δt_0. As seen from the track, the bullet flies for a time Δt and during this time covers a distance Δx. What is the relation between the pairs Δx_0, Δt_0 and Δx, Δt?

Let us begin with the length! In Newtonian physics it is obvious that $\Delta x = \Delta x_0 + V \cdot \Delta t$. How is this formula modified in relativity theory?

Due to Lorentz contraction, $\Delta x_0 \to \Delta x_0 \sqrt{1 - V^2/c^2}$, therefore $\Delta x = \Delta x_0 \sqrt{1 - V^2/c^2} + V \cdot \Delta t$, hence

$$\Delta x_0 = \frac{\Delta x - V \cdot \Delta t}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{54}$$

This formula expresses Δx_0 in terms of the pair $\Delta x, \Delta t$.

We must do the same for Δt_0. For this we will obviously need the speed of the bullet: relative to the train it is v_0, and relative to the track it is v: $v_0 = \frac{\Delta x_0}{\Delta t_0},\ v = \frac{\Delta x}{\Delta t}$. Using (54),

$$\Delta t_0 = \frac{\Delta x_0}{v_0} = \frac{\Delta x - V \cdot \Delta t}{v_0 \cdot \sqrt{1 - \frac{V^2}{c^2}}}. \tag{55}$$

Here we must substitute the relative velocity formula

$$v_0 = \frac{v - V}{1 - \frac{vV}{c^2}} = \frac{\Delta x - V \cdot \Delta t}{\Delta t - \frac{V}{c^2} \cdot \Delta x}. \tag{56}$$

The factor $(\Delta x - V \cdot \Delta t)$ can be canceled, therefore

$$\Delta t_0 = \frac{\Delta t - \frac{V}{c^2} \cdot \Delta x}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{57}$$

The pair of formulas (54) and (57) provides the solution of the problem. These are the Lorentz transformation formulas, which—because of their outstanding importance—we also state in a general form.

Let *K′* move with speed *V* relative to *K* along their common *x* -direction. Let two events occur whose coordinate and time differences in *K* are $\Delta x, \Delta y, \Delta z$ and Δt. Then the coordinate and time differences of the two events with respect to *K′* are determined by the Lorentz transformation formulas[27]

$$\Delta x' = \frac{\Delta x - V \cdot \Delta t}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad \Delta y' = \Delta y, \qquad \Delta z' = \Delta z \tag{58}$$

$$\Delta t' = \frac{\Delta t - \frac{V}{c^2} \cdot \Delta x}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{59}$$

The second and third equations of (58) express that—as we know from Chapter 7—the dimensions perpendicular to the direction of motion remain unchanged when transforming from *K* to *K′*.

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The Lorentz transformation is consistent with the invariance of proper time.

[27] The corresponding formula in Newtonian physics is the *Galilean transformation*

$$\Delta t' = \Delta t, \qquad \Delta x' = \Delta x - V \cdot \Delta t, \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z,$$

which is obtained from the Lorentz transformation in the limiting case $V \ll c$.

Let us assume that in *K′* the two events are connected[28] by a "clock" moving with constant velocity $\left(\frac{\Delta x'}{\Delta t'}, \frac{\Delta y'}{\Delta t'}, \frac{\Delta z'}{\Delta t'}\right)$. On this clock, between the two events, a proper time

$$\Delta\tau' = \Delta t' \cdot \sqrt{1 - \frac{1}{c^2}\left(\frac{\Delta x'}{\Delta t'}\right)^2 - \frac{1}{c^2}\left(\frac{\Delta y'}{\Delta t'}\right)^2 - \frac{1}{c^2}\left(\frac{\Delta z'}{\Delta t'}\right)^2} \tag{60}$$

elapses.

If, on the right-hand side, we express the primed differences in terms of the unprimed ones using the Lorentz transformations (58) and (59), then after a short calculation we arrive at

$$\Delta\tau'^2 = \Delta t'^2 - \frac{1}{c^2}\left(\Delta x'^2 + \Delta y'^2 + \Delta z'^2\right) = \Delta t^2 - \frac{1}{c^2}(\Delta x^2 + \Delta y^2 + \Delta z^2) = \Delta\tau^2, \tag{61}$$

which proves that proper-time intervals are independent of the choice of coordinate system. The "priming" of the proper time is therefore in fact unnecessary and misleading.

Equation (61) shows that the proper time elapsed on the projectile is invariant even if its trajectory is not purely along the *x* -direction. In fact, the validity of relation (61) is the most important criterion for the correctness of the transformations (58) and (59).

In proving (61) we did not make use of the fact that the quadratic expression

$$\Delta t^2 - \frac{1}{c^2}(\Delta x^2 + \Delta y^2 + \Delta z^2) \tag{62}$$

has a positive value, since we derived the Lorentz transformation for a timelike pair of events (imagining the firing and impact of a bullet). The invariance of this expression is therefore also valid when its value is zero or negative. When the sign is not positive, of course, (62) cannot be called a proper time. Therefore the name *square of the four-distance* was introduced for it—more precisely, for its multiple by c^2—together with the notation Δs^2:

$$\Delta s^2 = c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2). \tag{63}$$

Does the invariance of the square of the four-distance have a physical meaning even when it is not positive?

Yes, it does. The invariance of $\Delta s^2 = 0$ expresses the constancy of the speed of light:

$$c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = c^2\Delta t'^2 - (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) = 0. \tag{64}$$

In the special case, for example, when light propagates along the (x)-direction, i.e., $\Delta x = c \cdot \Delta t$, $\Delta y = \Delta z = 0$, then based on (58) and (59) we have

$$\Delta x' = \frac{c - V}{\sqrt{1 - \frac{V^2}{c^2}}}\Delta t, \qquad \Delta t' = \frac{1 - \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}}\Delta t, \qquad \Delta y' = \Delta z' = 0,$$

[28] For such a pair of events, we can always imagine (associate) a railway car and a bullet fired from a gun.

from which[29] $\Delta x' = c \cdot \Delta t'$.

In the case $\Delta s^2 < 0$, the invariance of the squared interval expresses the relativity of simultaneity. Indeed, if there exists an inertial frame (say K') in which the two events are simultaneous (($\Delta t' = 0$)), then Δs^2 is clearly negative. In all other inertial frames, $\Delta t \neq 0$.

For example, in Einstein's train thought experiment (Chapter 3), *K'* is the train and *K* is the rest frame of the embankment, with $\Delta t' = 0$, $\Delta x' = l_0$. Substituting these into the first equation of (58) and (59), then eliminating Δx from the two equations, after a short calculation we obtain

$$\Delta t = \frac{l_0 \cdot {}^{V}\!/_{c^2}}{\sqrt{1 - {}^{V^2}\!/_{c^2}}} \tag{65}$$

which, taking into account the Lorentz contraction (30), is identical to (7).

As we saw in Chapter 3, the relativity of simultaneity of two events does not lead to internal contradictions in the theory as long as the two events could only be connected by a clock moving faster than light. Since the condition $\Delta s^2 < 0$ corresponds to

$$\frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}{\Delta t} > c$$

this condition is indeed satisfied.

Based on the invariance of Δs^2, pairs of events can be unambiguously classified into one of three types:

$$\text{The pair } (\Delta t, \Delta x, \Delta y, \Delta z) \begin{cases} \text{timelike}, if\ \Delta s^2 > 0 \\ \text{lightlike}, if\ \Delta s^2 = 0 \\ \text{spacelike}, if\ \Delta s^2 < 0. \end{cases}$$

This terminology was already used in Chapter 3.

15. Comparison with the traditional way of introducing the Lorentz transformation.

The traditional introduction of special relativity can be regarded as *deductive*, while the introduction presented here may be considered *inductive*.

The traditional procedure can be broken down into the following steps.

1) After introducing the postulates, we immediately pose the question of what is the mathematical form of the transformation

$$\Delta t, \Delta x, \Delta y, \Delta z \longrightarrow \Delta t', \Delta x', \Delta y', \Delta z' \tag{66}$$

[29] When the direction of light propagation forms a nonzero angle with the *x*-axis, the direction of the velocity changes, but its magnitude remains *c*, as a consequence of (64).

that describes the transition between different inertial frames.
The transformation sought must satisfy the first postulate, which can be succinctly formulated as follows:

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0 \;\leftrightarrow c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 = 0. \qquad (67)$$

The equality on the left-hand side of the double arrow expresses that, in the inertial frame *K*, the pair of events is connected by a light signal (the pair is lightlike). But then a similar expression must also hold in *K'*, and conversely. The double arrow in (67) expresses this requirement.

Let us note that this is the same problem with which we started the previous chapter, except that instead of a pistol shot we now have a light signal.

2) Equation (67) can only be satisfied if the quadratic expressions on the two sides are proportional to each other:

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = F \cdot (c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2).$$

3) The proportionality factor F may depend on the relative velocity *V* of *K* and *K'* (in the form *V/c*), which we assume to point along the common *OX*, *O'X'* axis.
However, it can be shown that no such dependence exists and that *F=1*:

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2. \qquad (68)$$

4) The next task is to find the transformation that satisfies (68). From the homogeneity of space and time it follows that the transformation must be linear. First Einstein, and later—by a more elegant procedure—Minkowski, showed that the solution is given by the Lorentz transformations (58), (59).

5) Finally, from the Lorentz transformation we deduce the relation between proper time and coordinate time, the law of velocity addition, the Lorentz contraction formula, aberration, etc.

To derive the relation between proper time and coordinate time, for example, let us assume that a clock is at rest in *K'*: $\Delta x' = 0$. Then $\Delta x = V \cdot \Delta t$, and the elapsed time measured by the clock is called the proper time and is denoted by $\Delta\tau$. According to (59), therefore,

$$\Delta\tau \equiv \Delta t' = \Delta t \cdot \sqrt{1 - {V^2}/{c^2}}.$$

As we can see, as a consequence of the motion of the clock, the proper-time interval is smaller than the corresponding coordinate-time interval. One can, of course, point out that this relation can already be read directly from (68).

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The main advantage of the inductive line of reasoning followed in these lectures over the outlined deductive approach is that *it starts* by distinguishing conceptually between coordinate time and proper time. As we have seen, from this starting point one can reach the Lorentz transformation in a direct way through three mathematically very simple steps. Each step presents one aspect of the relativistic concept of time. In the first, we clarify the relation between the lengths of the two kinds of time intervals; in the second and third, we examine the effect of this relation on the transformation of velocity and on the size of objects in the direction of motion. A clear advantage of this method is that it naturally requires constant attention to the correct distinction between the two kinds of time, which helps in the gradual acquisition of their correct relativistic concept. Gaining this insight *retrospectively* is much more difficult.

For the sake of simplicity, the difference between the two lines of reasoning called inductive and deductive can also be described in another way. Special relativity is based on two fundamentally

novel theses. One is the constancy of the speed of light in different inertial frames, and the other is the breakdown of the unified concept of time into coordinate time and proper time. These are inseparably connected, but in teaching the theory it is unavoidable to choose which one to begin with, and thereby automatically which one will receive greater emphasis.

The traditional approach, called deductive, in accordance with the history of science, views relativity theory as the solution to the problem of the speed of light, an important tool of which is the correct synchronization of clocks. It derives the transformation of the spacetime coordinates of events from the requirement of the constancy of the speed of light—*before* introducing the two kinds of time. The inductive approach, by contrast, starts from the distinction between coordinate time and proper time and from establishing their relationship on the basis of the empirical constancy of the speed of light. For this purpose it must primarily deal with timelike pairs of events, and therefore it arrives at the Lorentz transformation not by means of lightlike, but by means of timelike event pairs.

Now, the timelike points of spacetime relative to the origin form a four-dimensional set, whereas the lightlike points form only a three-dimensional subset. For this reason, it is mathematically simpler in the inductive approach to find the Lorentz transformation valid for the entire spacetime.

16. The transformation of momentum and energy.

In the formulas of relativity theory, instead of the time variable *t* it is often advantageous to use the product *ct*, which has the dimension of length. If we rewrite the Lorentz transformations (58), (59) in the form

$$\Delta x' = \frac{\Delta x - \frac{V}{c}\cdot\Delta(ct)}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad \Delta y' = \Delta y, \qquad \Delta z' = \Delta z \tag{69}$$

$$\Delta(ct') = \frac{\Delta(ct) - \frac{V}{c}\cdot\Delta x}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{70}$$

then the coefficients of the transformation become dimensionless. This makes our formulas more transparent.

Let us now recall equation (53) (the four-momentum), written for simplicity in the form valid for uniform rectilinear motion:

$$\left(E/c\,, p_x, p_y, p_z\right) = m\left(\frac{\Delta(ct)}{\Delta\tau}, \frac{\Delta x}{\Delta\tau}, \frac{\Delta y}{\Delta\tau}, \frac{\Delta z}{\Delta\tau}\right) = \frac{m}{\Delta\tau}(\Delta(ct), \Delta x, \Delta y, \Delta z). \tag{71}$$

When we pass to a new, primed reference frame, the ratio $m/\Delta\tau$ remains unchanged, while the four quantities $(\Delta(ct), \Delta x, \Delta y, \Delta z)$ can be rewritten in terms of quantities in *K'* using the Lorentz transformation. Accordingly, in the transition $K \longrightarrow K'$, the total energy and the momentum can also be transformed into the new reference frame by applying the Lorentz transformations (69), (70):

$$p_x' = \frac{p_x - \frac{V}{c}\cdot\frac{E}{c}}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad p_y' = p_y, \qquad p_z' = p_z \tag{72}$$

$$\frac{E'}{c} = \frac{\frac{E}{c} - \frac{V}{c} \cdot p_x}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{73}$$

(The last equation is, of course, conveniently multiplied throughout by *c*.)

The right-hand side of (71) corresponds to a timelike interval between two points of a body moving uniformly along a straight-line trajectory; therefore

$$\Delta(ct)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s^2 = c^2 \Delta \tau^2.$$

Taking this into account, from (71) we obtain the important relation

$$E^2/_{c^2} - p_x^2 - p_y^2 - p_z^2 \equiv E^2/_{c^2} - p^2 = c^2 m^2 \tag{74}$$

between the total energy and the momentum.

Since $E = K + mc^2$, where K is the kinetic energy, it follows from this formula that

$$p^2 = 2Km + \frac{K^2}{c^2}.$$

In the nonrelativistic limiting case, the speed of light can be regarded as infinite, and thus we recover the Newtonian formula of classical physics for the kinetic energy expressed in terms of the momentum, $K = p^2/2m$.

17. Geometrical interpretation of Lorentz transformation.

Let us return for a moment to the problem that opens Chapter 14 and specify the origins of the coordinates and the time intervals. Let us place the origin O_0 of the x_0 coordinate axis fixed to the train at the rear end of the train, from where the pistol bullet is fired, and let us place the origin O of the x axis anywhere along the embankment. Furthermore, assume that when the two origins exactly coincide, the clocks $\mathcal{C}_0$ and $\mathcal{C}$, which are at rest at the respective origins and display coordinate time, both show zero time.

With this stipulation, the readings of the virtual clocks that display coordinate time and are at rest in the two coordinate systems are uniquely fixed, since these clocks are synchronized by light signals with the clock at rest at their own origin. If we make these refinements and fire the pistol at the instant $t = t_0 = 0$, then it is evident that the differences $\Delta x_0, \Delta t_0, \Delta x, \Delta t$ can be replaced by x_0, t_0, x, t, respectively.

In formulas (58) and (59), which are less closely tied to the introductory example, the Δ symbols can likewise be omitted if we specify the origins of the spatial and temporal coordinates in the same way. Along the common *X* axis, the systems *K* and *K'*, which move uniformly relative to each other, each carry with them a set of virtual clocks indicating the coordinate times *t* and *t'*, respectively. As a consequence of Einstein synchronization, two clocks that momentarily "overlap" show different coordinate times, and—as we have emphasized several times—it is impossible (and in fact unnecessary) to attempt to visualize continuously following and comparing the readings of all clock pairs that happen to occupy the same location at any given instant.

The most we can do is to prescribe, for a single selected pair of clocks that coincide at some moment, that at the instant of their coincidence they show, say, *t=t'=0*. The most convenient choice

is the pair of clocks that are at rest at the origins of the two coordinate systems. In this case, the event E_0, which in *K* has the coordinates $x_0=y_0=z_0=t_0=0$ will have the same, entirely zero-valued components in *K'* as well: $x_0'= y_0'= z_0'= t_0'=0$. If *E* is any other event with coordinates *x,y,z,t* and *x',y',z',t'*, then for these two events the Δ symbols in (58) and (59) can all be omitted, since, for example, $\Delta x = x-x_0'=x$. Thus we obtain

$$t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z. \tag{75}$$

The Lorentz transformation is most often written in this form.

Hermann Minkowski recognized that, beyond the kinematical meaning of transformation (75), an important geometrical interpretation can also be attributed to it, since it is very similar to the rotation of a spatial coordinate system

$$x' = x\cos\alpha - y\sin\alpha, \quad y' = y\cos\alpha + x\sin\alpha, \quad z' = z \tag{76}$$

about the *z* -axis.

In order to notice this similarity, the rotation angle α in (76) must be expressed—unusually—not in degrees or radians, but in units of length.

Let us imagine a sequence of right-angled triangles whose leg b is common (Fig. 11). Then the magnitude of the angle α opposite this leg can be characterized by the length a. In this case,

$$sin\,\alpha = \frac{a}{\sqrt{a^2+b^2}} = \frac{a/b}{\sqrt{1+\frac{a^2}{b^2}}}, \qquad cos\,\alpha = \frac{b}{\sqrt{a^2+b^2}} = \frac{1}{\sqrt{1+\frac{a^2}{b^2}}}.$$

With the help of these formulas, equation (76) takes the form

$$x' = \frac{x+\frac{a}{b}\cdot y}{\sqrt{1+\frac{a^2}{b^2}}}, \qquad y' = \frac{y-\frac{a}{b}\cdot x}{\sqrt{1+\frac{a^2}{b^2}}}, \qquad z' = z \tag{77}$$

which already bears a strong resemblance[30] to the Lorentz transformation (75). The angle α, which fixes the relative orientation of the coordinate systems, corresponds to the relative velocity *V* of the inertial frames, while the quantity *b*, the "universal constant," corresponds to the speed of light.

[30] In fact, Minkowski transformed (75) into a form similar to the much more transparent expression (76); however, because of the sign differences that appear throughout the formulas, this requires the use of hyperbolic trigonometry. For this reason, I chose instead to carry out the transformation in the opposite direction.

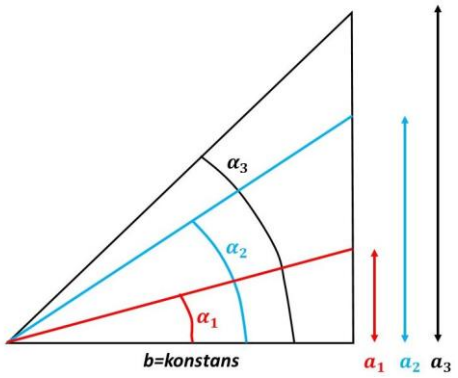

Figure 11

The analogy between rotation and the Lorentz transformation can also be established from another point of view. The distance of an arbitrary point from the origin is expressed by the same formula in *K'* as in *K*:

$$x'^2 + y'^2 = x^2 + y^2. \tag{78}$$

More briefly, this can be formulated as the statement

$$x^2 + y^2 = invariáns.$$

With the aid of (76), these formulas can easily be verified.

For the Lorentz transformation, the analogous formulas are the following:

$$\left.\begin{array}{c} c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 \\ c^2t^2 - x^2 - y^2 - z^2 = invariant. \end{array}\right\}$$

However, since $y' = y,$ and $z' = z$, it is often convenient to disregard these variables:

$$\left.\begin{array}{c} c^2t'^2 - x'^2 = c^2t^2 - x^2 \\ c^2t^2 - x^2 = invariant. \end{array}\right\} \tag{79}$$

***

The analogy we have pointed out here is also important because it helps us better understand why we speak of space and time in Newtonian physics, whereas in relativistic physics we speak of spacetime.

The difference between the two concepts can already be clearly illustrated using just two variables, *x* and *y*.

Let us imagine a horizontal *x* -axis and a vertical *y*-axis, and a curve, say, in the first quadrant. This diagram can in fact have two very different meanings.

It may express how the quantity *y* depends on *x*, that is, it may be the graph of the function $y = f(x)$. For example, *x* may represent the temperature *T* of some gas, and *y* its internal energy *U*. In this case, the lengths of the segments measured along the axes have physical meaning (e.g. a change in temperature), but the curve itself, as a geometric object, has no physical significance; the *length* of the curve representing the function $U = f(T)$ has no physical meaning. This curve is merely a visual way of representing a table containing the corresponding *T,U* pairs.

The situation is completely different when the curve represents, for example, a section of a road, and the *x,y* axes are two selected coordinate lines of an entire coordinate grid. In this case, the figure is not a graph but rather a map (and this is indeed what it is called). Only in the case of such a map does it make sense, for example, to rotate the coordinate axes about the origin. The curve, since it represents something that exists objectively, remains in place during this process, while its equation — the mathematical form of the function $y = f(x)$ — changes accordingly, because the unprimed variables must be expressed in terms of the primed ones using the inverse of (76).

Only in the second case can we speak of the *xy* -plane; in the first case it is more appropriate to speak simply of *x* and *y*.

When representing the motion of a point mass, we are in a more difficult situation, because the point moves in three-dimensional geometric space; therefore, a graph illustrating its trajectory

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

could only be “drawn” in four dimensions. Let us therefore restrict ourselves again to motion in the *x* -direction and pose the question: To which category does the curve representing the trajectory of the point mass in the *t,x* coordinate system belong? Is it a graph, or rather a map?

Let *A* and *B* be two points on the curve. The segment $\Delta t = t_B - t_A$ between them is equal to the elapsed time, and if the point mass does not reverse its direction of motion, then the segment $\Delta x = x_B - x_A$ is equal to the distance traveled.

However, the segment of the drawn curve itself between points *A* and *B* does not possess any meaningful length. This length would be given by the integral $\int_A^B \sqrt{dt^2 + dx^2}$which clearly contains a syntactic error: the two terms under the square root have different physical dimensions and therefore cannot be added to each other. For this reason, in Newtonian physics we do not speak of spacetime, but of space *x* and time separately: the diagram of our trajectory is not a map, but a graph.

In relativity theory, however, the proper time elapsed along the segment *AB* is very similar to a geometric length:

$$\tau_{AB} = \int_A^B \sqrt{dt^2 - \frac{1}{c^2}dx^2} = \int_A^B dt \sqrt{1 - \frac{v^2(t)}{c^2}}. \tag{80}$$

Apart from the negative sign, this is a formula similar to $\int_A^B \sqrt{dt^2 + dx^2}$, in which the consistency of physical dimensions is made possible by the existence of the universal constant *c*.

The essential point, however, is that τ_{AB} is a coordinate-system-independent invariant measure along the trajectories of point masses, just as geometric length is on ordinary maps. When we pass from one inertial frame to another using the Lorentz transformation (75), from a mathematical point of view we do the same thing as when we change the orientation of the coordinate grid on an ordinary map. In a Lorentz transformation, *x* and *t* "mix" with each other in a way similar to how *x* and *y* mix when rotating Cartesian coordinates. This is why, in relativity theory, we speak of spacetime instead of space and time.

Because of the differences in sign, the diagrams representing rotations and Lorentz transformations differ from each other (Figure 12). The diagram representing the Lorentz transformation is called a *Minkowski diagram*. The point *P* of the *xy*-plane corresponds, in spacetime, to the event *E*.

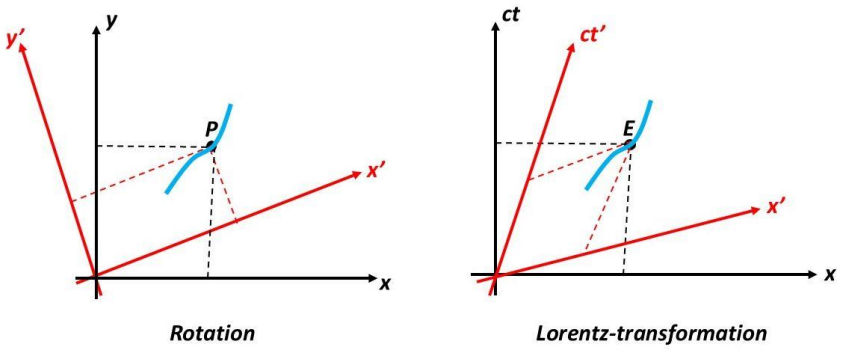

Figure 12

18. What can (should) be taught from all this?

Perhaps I have succeeded in convincing the Reader that the "conceptual gap" between special relativity and Newtonian physics is much deeper and wider than is generally thought. For this reason, in order to avoid serious misunderstandings that are difficult to correct, it would probably be best not to teach anything about relativity theory at the secondary-school level.

However, this cannot be done, because some students will certainly learn from other sources about at least three consequences of relativity theory: the unsurpassability of the speed of light, the mass–energy relation, and the twin paradox (and perhaps even Lorentz contraction). These are the topics that should be discussed in physics classes and extracurricular study groups. Point 5.1 of the current examination requirements at the advanced level essentially reflects this need:

"*Be familiar with the basic ideas of special relativity: the rejection of the concept of the ether, the limiting nature of the speed of light, the relativity of simultaneity, time dilation, length contraction, and mass–energy conservation*."

However, since I have no experience at all with teaching at the secondary-school level, I do not really know how this can be done well.

The mass–energy relation appears in point 6.2 of the DETAILED MATRICULATION EXAM REQUIREMENTS valid in 2022 under the title of mass defect, at both intermediate and advanced levels. It is important that students understand precisely the meaning, significance, and beauty of this law. For this reason, it should always be used in the form $E_0 = mc^2$ (as Einstein did in his

September 1905 paper), and students should be explained why the notation $E = mc^2$should be avoided. The compilers of the EXAM REQUIREMENTS were not aware of this either; otherwise, they would not have included in point 6.1 the requirement "*Be able to write down and interpret the relations for the mass and energy of a photon*," as well as the reference to mass conservation at the end of the quote from the previous paragraph.

In the matriculation exam requirements coming into effect in 2024, "photon mass" has been replaced[31] by photon momentum: "*Be able to write down and interpret the relations for the momentum and energy of a photon*." This is, of course, a completely correct formulation, but it would be better to formulate requirements that are connected with the essence of relativity theory, particularly the problem of time. The primary example here is time dilation, which is included in the requirements quoted above, and within that, the twin paradox.

To understand time dilation and the twin paradox, it is necessary to know the nature of the *proper time* τ, but this does not seem to be a particularly difficult concept. The precise meaning of *coordinate time*, however, probably goes far beyond the level that can be taught in secondary school. What is the point of dealing with the synchronization of clocks that do not even actually exist? It is particularly difficult for one to overcome the false impression that the constancy of the speed of light is somehow enforced by the tricky synchronization of imagined clocks.

Fortunately, in relativity theory, coordinate time serves the same function as the *t* used in high-school physics in accordance with every day experience, so proper time can be interpreted relative to this familiar notion of time without significantly distorting its meaning. If we accept this, then the formula $\Delta\tau = \Delta t \cdot \sqrt{1 - v^2/c^2}$ can be written down and explained, and perhaps even derived using the optical Doppler effect. It may also be advisable to note that, when we want to emphasize that we are not talking about proper time but the familiar *t*, we can refer to it as "coordinate time," since *t* is what is measured along the horizontal coordinate axis of graphs representing the motion of bodies.

In connection with the unattainability of the speed of light, we somehow need to explain why a rocket that burns the same amount of fuel per unit of time cannot exceed the speed of light[32]. We have seen that this is partly due to time dilation: a unit of proper time on the spacecraft corresponds to a longer Δt in the reference frame the faster the spacecraft moves. Therefore, if we plot the trajectory of such a spacecraft on a graph, its acceleration will decrease progressively. Perhaps it is not a serious error to omit that there is another reason as well (the relativistic form of the velocity addition law).

Finally — and perhaps I should have started with this — the explanation of the constancy of the speed of light should be based, instead of the Michelson–Morley experiment, on the concrete operational statement that if we measured the speed of light in inertial frames moving at different velocities (for example, on railway cars) using any previously learned method, we would always obtain the same value *c=300 000 km/s*. This requires no imagined clock synchronization.

The problem is that the constancy of the speed of light in this form is not striking enough. Perhaps the discussion at the end of Chapter 2 (on page 8) makes it clear why no one seriously considered this possibility before Einstein, and why it is nevertheless so surprising.

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[31] In the earlier version of the requirements, even the velocity dependence of mass had to be "known" by students taking the advanced-level school-leaving exam.

[32] We may neglect the fact that the mass of the rocket decreases in the meantime. If the fuel is particularly efficient, even a small amount of it is sufficient to reach very high speeds.

Appendix: The train-tunnel paradox

A train of proper length l_0 passes through a tunnel of the same proper length at speed V. In the rest frame of the track (the tunnel), the train is shorter than the tunnel. In the rest frame of the train, however, the tunnel is shorter than the train. Is this correct?

Yes, it is correct.

Is there a contradiction between the two statements?

No, because we are not making contradictory statements about the result of the same observation.

How do we determine the two lengths?

- The length of the moving train is measured by the observer standing on the track by multiplying the time it takes for the train to pass by by the speed of the train.
- The length of the tunnel is measured by the observer on the train by multiplying the time the train spends inside the tunnel by the speed of the tunnel (relative to the train).

Both observers obtain the same number, which is smaller than l_0. There is no contradiction in this.

But now let's twist the problem a bit:

The train is carrying treasure, and a gang wants to rob it. The gang leader has studied relativity and knows that the moving train is shorter than the tunnel. Therefore, the gang secretly installs strong, gate-like barriers at both entrances of the tunnel, which can be closed by a radio signal.

One gang member positions himself at a point equidistant from the two ends of the tunnel, with the signal transmitter, and receives the command from the gang leader that when the train disappears inside the tunnel, he should activate the barriers. The rest of the gang then captures the train trapped in the tunnel and robs it.

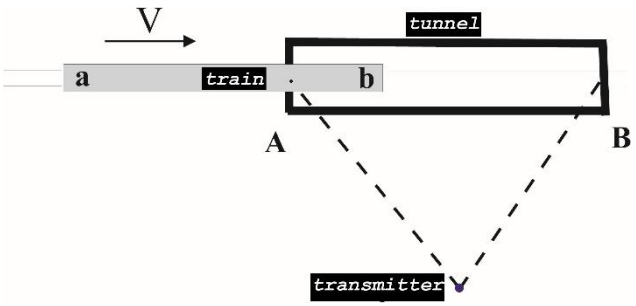

Figure 13

After the train departs, the secret service somehow gets wind of the planned raid and instructs the train driver via mobile phone to stop the train immediately. However, the train driver was good at physics and reassures the secret service agent: The tunnel undergoes Lorentz contraction relative to the train, so at least one end of the train will definitely stick out of the tunnel. Therefore, the gang member watching at the signal transmitter will not have a chance to activate the barriers.

Who is correct, the gang leader or the train driver? Could it be that both are right?

No, it cannot be, because for the same event (the closing of the gates), two mutually exclusive statements cannot both be true.

How would the gang leader argue?

The length of the tunnel is l_0, and the train's length is $l_0\sqrt{1-V^2/c^2}$; therefore, the train remains inside the tunnel for a time $\left(l_0 - l_0\sqrt{1-V^2/c^2}\right)/V$. Our accomplice thus has enough time to activate the signal transmitter. Is this explanation acceptable? Clearly, yes.

Then the train driver was mistaken. Is there a conspicuously uncertain point in his reasoning? Yes, there is. He did not consider that *in the train's rest frame K', the gates do not close simultaneously*.

Which gate closes later?

In *K'*, the train is at rest and the tunnel (with the signal transmitter) moves from right to left. Since the event in the direction of motion occurs later (see the example in Chapter 3), the left gate *A* closes later.

When the already-closed gate at the *B* end of the tunnel collides with the front of the train, how much of the train still sticks out of the tunnel? The difference between the train's length and the contracted tunnel length, that is, $l_0 - l_0\sqrt{1-V^2/c^2}$ meters.

Then the train driver was right after all! When the barrier at the *B* end of the tunnel collided with the front *b* of the train, it immediately began pushing the train along, so that $l_0 - l_0\sqrt{1-V^2/c^2}$ meters of the train continued to stick out of the tunnel. The gang's agent could have activated the signal transmitter only when the train was completely inside the tunnel. But according to this, no such moment exists. Is this reasoning correct?

No, because the effect of the train's front colliding with the *B* gate reaches the rear of the train only after a finite time, even according to Newtonian physics. On a human scale this time is imperceptibly small, but due to the (supplemented) 1st postulate, it is certainly larger than l_0/c. Only after this does the whole train begin to move left along with the tunnel. And this delay may be enough for the left-moving tunnel to engulf the still-stationary rear end of the train. Is it really enough?

Yes, because it takes the *A* starting point of the tunnel a time $\left(l_0 - l_0\sqrt{1-V^2/c^2}\right)/V$ to reach the rear end of the train (that is, to engulf the train), and this time is shorter than l_0/c.

Let us justify this statement!

$$\left(l_0 - l_0\sqrt{1-V^2/c^2}\right)/V \overset{?}{<} l_0/c$$

$$1 - \sqrt{1-V^2/c^2} \overset{?}{<} V/c$$

$$1 - V/c \overset{?}{<} \sqrt{1-V^2/c^2}$$

$$\sqrt{1-V/c} < \sqrt{1+V/c}$$

which is obviously true.

But does the *A* gate not close onto the end of the train in the meantime? What is the condition for this not to happen?

Let E_a and E_b be the two events when the appropriate ends of the train and the tunnel meet each other: the *a* end of the train with the *A* end of the tunnel, and the *b* end of the train with the *B* end of the tunnel, respectively. The gang member can activate the signal at any time between the two events. Let us first assume that he activates it immediately before E_b (that is, just before the reappearance of the train). What is the condition for the tunnel to still swallow the (a) end of the train before the closing of gate *A*?

That the time interval $\Delta t'$ between the closing of the gates be greater than $\left(l_0 - l_0\sqrt{1 - V^2/c^2}\right)/V$. How can we check whether this condition is satisfied?

According to (65), $\Delta t' = \frac{V \cdot l_0}{c^2\sqrt{1-V^2/c^2}}$ therefore the question is the following:

$$\frac{V \cdot l_0}{c^2\sqrt{1 - \frac{V^2}{c^2}}} \overset{?}{\leqq} \frac{\left(l_0 - l_0\sqrt{1 - \frac{V^2}{c^2}}\right)}{V} \tag{81}$$

$$\frac{V^2/c^2}{\sqrt{1 - V^2/c^2}} \overset{?}{\leqq} 1 - \sqrt{1 - \frac{V^2}{c^2}} = \frac{V^2/c^2}{1 + \sqrt{1 - V^2/c^{"}}}$$

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and this is satisfied, because the denominator on the right-hand side is larger than that on the left-hand side. Thus, the train is trapped in the tunnel also from the viewpoint of frame *K'*.

Now let us consider the case when the signal transmitter is activated immediately after the disappearance of the train (that is, practically simultaneously with E_a. What happens then when viewed from the rest frame of the train?

As soon as the front end *A* of the tunnel passes the *a* endpoint of the train, its barrier closes. If there were no barrier at the *B* end of the tunnel, then due to Lorentz contraction of the tunnel, a length of $l_0 - l_0\sqrt{1 - V^2/c^2}$ of the train would stick out of the tunnel. But this cannot happen, because the *B* -barrier had already closed $\Delta t' = \frac{V \cdot l_0}{c^2\sqrt{1-V^2/c^2}}$ seconds *earlier*, and as a consequence of (81) the *b* front of the train was still inside the tunnel at that moment. Therefore the train again remains trapped in the tunnel, and obviously this is what happens as well when the signal transmitter is activated at any time between E_a and E_b.

(Translation by Open Ai)