Quantum Theory within the Framework of General Relativity. A Local Approach[1]

Peter Hraskó[2]

Abstract: A local conception (in the sense of the equivalence principle) is proposed to reconcile quantum theory with general relativity, which allows one to avoid some difficulties — as e.g. vacuum catastrophe — of the global approach. All nonlocal aspects of quantum theory, including EPR paradox, remain intact.

1.Introduction

Quantum theory on a given space-time has been the subject of intensive study over the past decades[(1)]. The announcement by S.W.Hawking of the thermal radiation of black holes[(2)] and the analysis of the behaviour of accelerated radiation detectors by W.G.Unruh[(3)] are outsanding instances of this research.

The quantization procedure adopted in these works was based on the positive and negative frequency modes extended over Cauchy surfaces across space-time as a whole. Due to this basic feature this approach may be characterized as *global*. The greatest difficulty of this field of research consists in the complete lack of relevant experimental evidences so that it must be carried out deductively on purely theoretical grounds. Therefore, at present we have no definitive observational facts which could corroborate or refute the basic principles of the global theory.

The purpose of the present work is to call attention to an alternative way to deal with the same subject which is motivated mainly by some difficulties of the global approach. The foremost problem we have in mind is the cosmological constant problem (or vacuum catastrophe) which has already a long history[(4)]. It consists in the enormous vacuum energy of particle quantum fields in the curved space-time around the Sun which should have led to the complete distortion of the orbits of the planets. A less known difficulty is connected with the frequency mixing in geodesic motion in curved space-

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[2]Department of Theoretical Physics, Janus Pannonius University, Pécs, Ifjuság u. 6, 7604 Pécs, Hungary. E-mail: peter@hrasko.com

time. The term "frequency mixing" expresses the fact that the negative frequency modes present in the vacuum of a quantum field acquire positive frequency components as seen by a moving body. It is this phenomenon which is the origin of the Unruh-effect: Photon detectors accelerated in the empty pseudo-Euclidean space-time of special relativity are expected to be permanently excited while continuously emitting photons. In a curved space-time as e.g. in that around the Sun frequency mixing may occur even for a non-accelerated (freely falling) detector. Though the rate of this effect is far too small to be observable, its existence acquires principal importance if considered in the light of the Einsteinian principle of equivalence.

As it is well known this principle asserts(5) that bodies of reference, falling freely without rotation, are the only true realizations of inertial frames in Nature. They possess all the basic properties of the inertial frames of reference introduced in special relativity with a single exception — they are of finite rather than infinite extension (locality vs. globality). A freely orbiting spacecraft if it doesn't rotate is the simplest example of a body of reference, carrying a local inertial frame of Einsteinian sense. If bodies moving freely on geodesics experience indeed frequency mixing then a photon detector which is at rest in such a spacecraft will be continuously emitting photons. This is, however, in sharp contradiction with the principle of equivalence which requires that physical phenomena in local inertial frames take place just as in the inertial frames of special relativity. In particular, photon detectors at rest in vacuum must remain — after, perhaps, some relaxion — in their ground state.

Both the vacuum catastrophe and the frequency mixing in geodesic motion are connected with the infinite extension of the modes in space-time on which the global quantization procedure rests. The approach to be outlined in the following sections will be *local* in the sense that it requires global modes characteristic to special relativity which are insensitive to the curvature of space-time and correspond more closely to the spirit of the equivalence principle. The term "local" does not, therefore, imply any change in the nonlocal features of quantum theory. It refers to the way quantum theory is reconciled with the space-time of general relativity.

2.The two types of coordinates. Freely falling bodies of reference.

It is a well known fact that spinor fields can be defined only in (pseudo)-

orthogonal coordinates[6] since spinor transformation rule cannot be generalized from (hyperbolic) rotation to general coordinate transformations. As a consequence, on curved space-time spinors can be described only with respect to some orthonormal tetrad field. In spite of this complication spinors play a highly important role in quantum physics. This fact leads to the intriguing suspicion that, perhaps, orthonormal tetrads may be more significant for quantum theory in curved space-time than just being an auxiliary device.

On the world line $\mathcal{L}$ of every massive object an orthonormal tetrad is naturally defined by Fermi-Walker transport (see the Appendix) which for geodesics reduces to parallel transportation. The time-like element of the tetrad coincides with the unit tangent vector to the world line. This tetrad field will serve as the starting point for the construction of two different types of coordinates.

The first of them are the Fermi-coordinates attached to $\mathcal{L}$. To obtain the Fermi coordinate-time t of an event E one has to drop a perpendicular to $\mathcal{L}$ from E and identify t with the proper time τ of the point P of intersection of the two curves. The perpendicular PE is a space-like geodesic whose 3-direction at P with respect to the tetrad at that point is given by the polar angles ϑ, φ. Then the Fermi space-coordinates x, y, z of E are equal to

$$x = l \cdot \sin\vartheta \cos\varphi, \qquad y = l \cdot \sin\vartheta \sin\varphi, \qquad z = l \cdot \cos\vartheta,$$

where l is the geodesic distance between P and E.

In this section we confine ourselves to the case when the body of reference performs free geodesic motion, i.e. its world line is the geodesic $\mathcal{G}$. The main virtue of the Fermi-coordinates is that, using them, the metric in this case becomes Minkowskian ($g_{ij}(\mathcal{G}) = \eta_{ij}$) and the connection coefficients vanish ($\Gamma^i_{jk}(\mathcal{G}) = 0$) *along the geodesic* $\mathcal{G}$[7][8]. Therefore, the Fermi-coordinates are the generalizations of the Minkowski-coordinates in the global inertial frames of special relativity to the local inertial frames of general relativity. It is the Fermi-coordinates which are naturally employed by the astronauts on a freely orbiting nonrotating spacecraft, assuming their abode to be at rest and their clocks to show the time. Though the relations $g_{ij} = \eta_{ij}$ and $\Gamma^i_{jk} = 0$ are — strictly speaking — valid only in a single point within the spacecraft they remain practically valid in some more or less extended neighborhood N of it[3]. It is this N where our physical theories must be valid in their purest form

[3]This neighborhood is the same, where tidal forces are negligible.

established for the global inertial frames of special relativity in Minkowski-coordinates. In particular, the experiments in quantum physics which can be performed within N must give results in conformity with the theoretical predictions made in special relativity.

The second type of coordinates will be called *parametric*. They will be assumed numerically equal to the Fermi-coordinates but the manifold which they parametrize is of pseudo-Euclidean structure whose metric in parametric coordinates is Minkowskian. The points of the parametric space may be identified[4] with the vectors of the tangent space to space-time at the point $\tau = 0$ of $\mathcal{G}$. Therefore, this tangent space endowed with pseudo-Euclidean structure may be taken to represent the parametric space. Parametric coordinates cover the whole of the tangent space while Fermi-coordinates cover only part of the space-time. The important point is that in the neighborhood N the metrical properties of the space-time do not discriminate between the space-time Fermi-coordinates and the parametric coordinates.

We are now prepared to formulate the basic assumption of our local approach: We assume that *the vectors of the quantum theoretical Hilbert-space are labelled by parametric rather than space-time coordinates.* In other words the coordinates in the Schrödinger-equation are parametric coordinates. This means that quantum theory works in the tangent space though the phenomena it describes take place, of course, in space-time.

Let us illustrate the interplay of the two types of coordinates by the calculation and interpretation of the emission of a photon by an atom which is at rest in the reference body moving on $\mathcal{G}$.

The calculation of the decay probability is performed in parametric coordinates in exactly the same way as in special relativity, because the parametric space is by definition pseudo-Euclidean and the metric induced in it by the Fermi-coordinates is Minkowskian. The decay constant given by this calculation may slightly differ from that which can be obtained by the space-time quantum electrodynamics of the global approach since the modes in the latter case reflect the curvature of the space-time to which the parametric space is insensitive. Were the difference sufficiently large the comparison of the calculated decay constants with observation would permit us to choose between the two approaches. But even if the numerical disagreement is very small it reflects the principal difference between the global and local points

[4]This identification is not obligatory but offers a useful visualization of the parametric space.

of view. The space-time modes in terms of which the global approach is formulated are "realistic" in the sense that they reflect the curvature of the far regions of space-time as if they were in some way really there. The local point of view, on the contrary, assumes that submicroscopic phenomena, such as the decay of an excited atom, are truly local effects whose rates are fully determined by local environment. If this idea is correct it would be a misinterpretation to assign the modes occurring in these calculations to space-time. I stress that this inclination toward local description is suggested — if not dictated — by the principle of equivalence to which, as explained in the Introduction, the global approach seems to contradict.

In parametric space the rays of the emitted light are zero geodesics. The identification of the parametric coordinates with the space-time Fermi-coordinates permits us to map these rays into space-time. But this mapping does not in general conserve the geodesic nature of world lines. So, concerning light deflection in gravitational field, we arrive at a contradiction with the experience since the explanation of light deflection is based on the fact that light rays are null geodesics in space-time. Therefore, the relegation of quantum theory to the tangent space leads to the unavoidable consequence that its predictions which are not strictly microscopic become increasingly worse when moving off the world line of the body of reference whose parametric space is the terrain of the calculations.

The proper treatment of this phenomenon is made possible by the fact just mentioned that the photons travelling in space between distant bodies of reference follow quasiclassical trajectories, i.e. move on zero geodesics. In the calculation of their emission they appear as the out states $|k_1, \epsilon_1; out\rangle$ of definite momentum and polarization. The momentum is defined by the tangent to the trajectory at its initial point with respect to the reference frame 1 where the emission takes place. Similarly, the absorption of the photon on the body 2 takes place from the in-state $|k_2, \epsilon_2; in\rangle$. The "propagation matrix" ${}_2\langle k_2, \epsilon_2; in|k_1, \epsilon_1; out\rangle_1$ is determined by the quasiclassical motion of the photons between the reference bodies 1 and 2 in conformity with the fact that our knowledge of both light deflection and red shift which are the paradigms of the phenomena under consideration comes entirely from the classical theory of light propagation. The role of quantum theory in these processes is confined to the determination of the emission rates and absorption cross sections.

At the conclusion of this section we add on more argument in favor of parametric space quantization to those brought forward in the Introduction.

It is well known that canonical quantization leads to a theory consistent with experiment only if it is performed in terms of Cartesian coordinates and momenta. In spherical coordinates, for example, commutation relations $[\hat{\varphi}, \hat{p}_\varphi] = i\hbar$ etc. are easily satisfied by $\hat{\varphi} = \varphi$ etc. and momentum operators

$$\hat{p}_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} - \frac{2}{r}\right), \qquad \hat{p}_\vartheta = \frac{\hbar}{i}\left(\frac{\partial}{\partial \vartheta} - \cot\vartheta\right), \qquad \hat{p}_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial \varphi},$$

which are Hermitean (more precisely: symmetrical). If, however, one proceeds further and constructs the Hamiltonian of the hydrogen atom out of r, ϑ, φ, $\hat{p}_r$, $\hat{p}_\vartheta$, $\hat{p}_\varphi$, then one obtains an operator whose spectrum differs from that known from quantum mechanics and is, of course, in contradiction with experiment. One is, therefore, *compelled* to quantize in Cartesian coordinates[5], transformation to curvilinear coordinates being allowed only afterwards. In Euclidean space this restriction does not present any problem since in this space the set of Cartesian coordinates is preferred (it can be selected unambiguously and equivalence within the set is ensured). Since, however, in general pseudo-Riemannian space-time no preferred coordinates of this type exist, this manifold seems to present an unfriendly environment for canonical quantization.

As we have seen, Fermi-coordinates attached to a time-like world line are preferred in the same sense as Cartesian coordinates in Euclidean space so it is geometrically meaningful to prescribe that quantization be performed in terms of them. If, in addition, they are considered as Minkowski-coordinates in a pseudo-Euclidean parametric space canonical quantization becomes unambiguously defined. So far this has been established only for parametric spaces attached to geodesics but in the next section this point of view will be extended to a class of accelerating bodies of reference.

3.The two types of coordinates. Accelerating bodies of reference.

Let us consider the case when the body of reference under consideration is accelerating, i.e. moves on a general time-like world line $\mathcal{L}$ which is not geodesic. The tetrad field along $\mathcal{L}$ is obtained by Fermi -Walker transport rather than parallel transport. The same kind of reasoning which for

[5]"This assumption [of replacing classical canonical coordinates by corresponding operators] is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a Cartesian system of axes and not to more general curvilinear coordinates."[(9)]

a geodesic $\mathcal{G}$ leads to the relations $g_{ij}(\mathcal{G}) = \eta_{ij}$, $\Gamma^i_{jk}(\mathcal{G}) = 0$ permits us to determine $g_{ij}(\mathcal{L})$ and $\Gamma^i_{jk}(\mathcal{L})$ — or $\left(\frac{\partial g_{ij}}{\partial x^k}\right)_{\mathcal{L}}$ — again (see the Appendix). Then, up to first order in Fermi coordinates x, y, z the metric tensor is given by the relation

$$g_{ij} = \eta_{ij} + \frac{2}{c^2}(\vec{w} \cdot \vec{r}) \cdot \delta_{i0}\delta_{j0}, \tag{1}$$

where $x^0, x^1, x^2, x^3 \equiv ct, x, y, z$, $\vec{r} = (x, y, z)$, and $\vec{w}$ in general depends on τ which is identical to the Fermi — or, rather, Fermi-Walker — coordinate time.

Let us confine ourselves to the most important case of constant acceleration when in Fermi-coordinates $w^i = constant$ along $\mathcal{L}$. For a laboratory at rest on the Earth this formula gives $g_{ij} = \eta_{ij}$ except for $g_{00} = 1 + \frac{2\Phi}{c^2}$ where $\Phi = gz$ is the gravitational potential in the vicinity of the Erth's surface.

Parametric coordinates are numerically equal to the Fermi coordinates and in a neighbourhood N of $\mathcal{L}$ even their metric properties coincide in these coordinates. This coincidence ensures the applicability of the results of parametric space quantum theoretical calculations to space-time within local frames. On the other hand, as we have stressed in the preceding section, the structure of the parametric space must be pseudo-Euclidean. Therefore, in Fermi coordinates the fundamental quadratic form $ds^2 = F(z) \cdot c^2dt^2 - dx^2 - dy^2 - dz^2$ in this space must be such, that the Riemann-tensor vanish and at $z = 0$ the relations $F = 1$ and $\frac{dF}{dz} = \frac{2g}{c^2}$ fulfill. These conditions have the unique solution $F(z) = \left(1 + \frac{gz}{c^2}\right)^2$ (see the Appendix). The metric in $ds^2 = \left(1 + \frac{gz}{c^2}\right)^2 c^2dt^2 - dz^2$ is identical to the Rindler metric[(10)] with the coordinate singularity shifted from $z = 0$ to $z = -c^2/g$. This quadratic form is transformed to Minkowskian (primed) coordinates by the formulae

$$\begin{aligned} ct' &= \frac{c^2}{g}\left(1 + \frac{g}{c^2}z\right)\sinh\frac{gt}{c} \\ z' &= \frac{c^2}{g}\left[\left(1 + \frac{g}{c^2}z\right)\cosh\frac{gt}{c} - 1\right], \end{aligned} \tag{2}$$

which must be supplemented by $x' = x$, $y' = y$.

When the presence of gravitation cannot be neglected quantization must be performed in these primed coordinates. An investigation of this kind is the COW experiment[11] in which the effect of the weight of the neutron on the interference pattern was studied. In primed coordinates the Schrödinger equation is the one for a free neutron:

$$i\hbar\frac{\partial\psi'}{\partial t'} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi'}{\partial x'^2} + \frac{\partial^2\psi'}{\partial y'^2} + \frac{\partial^2\psi'}{\partial z'^2}\right). \tag{3}$$

Since the observations in space-time are expressed in terms of the (unprimed) Fermi-coordinates it is expedient to transform this equation to unprimed coordinates. In this nonrelativistic case the $c \longrightarrow \infty$ form of (2) is relevant:

$$t' = t, \qquad x' = x, \qquad y' = y, \qquad z' = \frac{1}{2}gt^2.$$

Then, introducing $\psi(x, y, z, t)$ instead of $\psi'(x', y', z', t')$ by the relation

$$\psi = \psi' \cdot e^{-ig\left(\frac{m}{\hbar}tz + \frac{mg}{6\hbar}t^3\right)} \tag{4}$$

we transform (3) to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + mgz \cdot \psi.$$

The effect of the last term on the interference pattern was indeed observed.

Consider now the Pound-Rebka experiment[12] in which the existence of the gravitational red shift was demonstrated in the laboratory. From the principal point of view two distinct laboratories were involved in the experiment, one above the other and both at rest in the gravitational field of the Earth. As we have already emphasized the red shift itself is a purely classical phenomenon which is contained in the propagation matrix between the two laboratories. The role of quantum electrodynamics is limited to the treatment of the emission in the lower and absorption in the upper laboratory (or vice versa) both of which were Mössbauer-transitions. Though the influence of terrestrial gravitation on these processes is certainly negligible it is a principal question how to calculate them in our local approach.

The first step is to formulate quantum field theory in the parameter space attached to either laboratory as body of reference, accelerating upward with constant acceleration. Just as in the case of the Schrödinger-equation discussed above the field quantization also has to be performed in the primed

Minkowski-coordinates. In particular, the ground state must be identified with the vacuum in the primed description (primed vacuum). Having this done, the transcription to unprimed Fermi-coordinates must follow with the aid of an appropriate unitary transformation — the analogue of (4), — since the physical conditions are given in terms of the unprimed coordinates: the proper time is t rather than t' and the radiating and absorbing atom is at rest with respect to the coordinates x, y, z rather than x', y', z'.

For a scalar field the unitary transformation involved here has been dealt with earlier in great detail (Ref.1. Section 4.5). The motivation for this study was to illustrate the peculiarities of field quantization in curved space-time on an example which allowed the comparison of quantization in Minkowski and curvilinear coordinates. The unitary transformation was found to have the form of a Bogoliubov-transformation of the primed emission and absorption operators. The primed vacuum which is the ground state in our approach contains particles which can be absorbed by the unprimed absorption operators. As a result, the detectors which are at rest in the gravitational field of the Earth are expected to continuously emit particles, though the rate of this effect is very low.

The mathematics behind this phenomenon is precisely the same which led to the recognition of the Unruh-effect, the radiation of detectors accelerated in the space-time of special relativity. The context is, however, quite different. First, the scene is parametric space rather than space-time. Second, the approach outlined here leads to the radiation of detectors which are at rest in the static gravitational field of the Earth, i.e. accelerating in the sense of performing non-geodesic motion. When, on the other hand, this same situation is described in the global approach, choosing a procedure based on Schwarzschild-time (static quantization), then one obtains that detectors are nonradiating when at rest in Schwarzschild-coordinates — but in general radiate when moving on geodesics (i.e. at rest in local inertial frames). We note finally that the Unruh-effect follows from our approach too.

4.Compatibility with EPR

The fundamental nonlocal nature of quantum mechanics is most clearly expressed in the EPR-type correlations violating Bell-inequalities[(13)(14)]. The observation of this phenomenon requires three macroscopic instruments: the source of the correlated pairs of — say — photons and the two detectors wich measure their directions of polarization[(15)]. These instruments may belong to different bodies of reference, locating at arbitrarily large distances from each other. The natural question arises: what the local approach can tell us on the correlations under such unusual circumstances?

For the description of the phenomenon let us choose the parameter space attached to the source of the photon pairs. Then the description of the source and the pair will be undisturbed by the metrical incongruence of the parameter space and space-time. On the other hand, the detectors may be located in such domains of space-time, where this incongruence is already significant[6] and, therefore, their properties may appear distorted as described in the parameter space of the source.

We have already pointed out an example of such distortions: the space-time image of the parametric space quasiclassical trajectories are, in general, different from the true space-time trajectories. As another example consider a photon detector, moving freely on a space-time geodesic. In the parameter space of a body of reference which is sufficiently far from this detector it will appear accelerated and, therefore, continuously counting photons. That would be, however, a false conclusion since this same detector described in its own parameter space will permanently be in its ground state.

Returning to the space-time trajectory of photons, we observe that though these trajectories deviate from those given by the parametric space quasiclassical calculation the more the farther they are from their source, the helicities, being invariant on the geodesics in both space-time and parameter space, do not experience distortions at all. Therefore, as far as the helicities are concerned, calculations performed in the parameter space of the source lead to reliable conclusions even for faraway detectors. Since the quantum theory in the parameter space of a given world line is just that quantum theory which we know from textbooks, the distance between the detectors and the source does not influence the correlation of the polarizations even if they enter into disconnected bodies of reference. This is a bit disappointing conclusion since a local approach might have expected to give different results for EPR type experiments within laboratory and on cosmic scale which could in principle be used to choose between them. However, the version of the theory, pursued in the present work, seems not to offer us this opportunity.

[6]We confine our discussion to that part of the space-time which is covered by the Fermi-coordinates of the world line of the source. Questions, concerning the extension of this domain, are left for future study.

5.Summary

Our accepted manner to describe physical reality does not assign decisive role to the perspective — or standpoint — of the observer who is the subject of the description. This is true even when the world does not seem the same from different points of observation as e.g. in the realm of the cosmological principle. For example, the Solar System can be correctly described from both the standpoint of the Sun and the Earth since the superiority of the heliocentric system consists certainly not in the reliability of the former and unreliability of the latter description.

As a matter of fact, the version of the local approach outlined in the present work offers an alternative role to the point of observation as far as the phenomena of quantum physics are concerned. The *possibility* of description remains independent of the standpoint but its *validity* becomes depending on it in a significant way. But since observation can in principle be performed from any macroscopic body of reference no sensible questions remain necessarily unanswered by this narrowing of the horizon. If they do the local approach must be abandoned.

In the limit of flat space-time our local approach becomes identical with the usual space-time form of quantum theory, since in this case it is possible to identify the parameter spaces with the whole of the space-time. In this limit viewing the world from different bodies of reference gives equally valid descriptions which differ from each other solely by the choice of the coordinate systems.

Appendix

The Fermi-Walker transport of a vector V^i along a time-like world line $\mathcal{L}$ is given by the formula

$$\frac{DV^i}{d\tau} = \frac{1}{c^2} g_{kl}(w^i u^k - u^i w^k) V^l, \tag{A1}$$

in which u^i is the four-velocity ($u \cdot u = c^2$), $w^i = \dfrac{Du^i}{d\tau}$ is the four-acceleration and $\dfrac{D}{d\tau}$ the invariant derivative along $\mathcal{L}$:

$$\frac{DV^i}{d\tau} = \frac{dV^i}{d\tau} + \Gamma^i_{kj} V^j u^k. \tag{A2}$$

The defining properties of the Fermi-Walker transport are:

1. the tangent vectors are transported into each-other,
2. the scalar product is left invariant and
3. for a geodesic ($w^i = 0$) it reduces to parallel transport (absence of rotation).

Let us choose Fermi-coordinates (Section 2) attached to $\mathcal{L}$ and consider the space-like geodesics, connecting E and P, given by the formulae

$$t = constant, \qquad x = l \cdot n^x, \qquad y = l \cdot n^y, \qquad z = l \cdot n^z,$$
$$n^x = \cos\vartheta \cos\varphi, \qquad n^y = \cos\vartheta \sin\varphi, \qquad n^z = \sin\vartheta,$$

ϑ and φ being the polar angles of the geodesic at P. These expressions satisfy the geodesic equation

$$\frac{d^2x^i}{dl^2} + \Gamma^i_{jk}\frac{dx^j}{dl}\frac{dx^k}{dl} = 0,$$

which leads to $\Gamma^i_{\mu\nu} n^\mu n^\nu = 0$ ($i = ct, x, y, z;\ \mu, \nu = x, y, z$). Since at P n^μ may point in any 3-direction, we obtain for a symmetric connection

$$\Gamma^i_{\mu\nu}(\mathcal{L}) = 0.$$

Consider now the elements of the tetrad field along $\mathcal{L}$ whose components are $e^i_{(m)} = \delta^i_m$. According to the definition of the Fermi-coordinates they satisfy (A1):

$$\Gamma^i_{kj} \cdot \delta^j_m u^k = \frac{1}{c^2} g_{kl}(w^i u^k - u^i w^k)\delta^l_m. \tag{A3}$$

The construction of the Fermi-coordinates also ensure that along the coordinate lines which cross at P the coordinates are measured by the length (or proper time) on them. From this and from the pseudo-orthonormality of the tetrads it follows that

$$g_{ij}(\mathcal{L}) = \eta_{ij}. \tag{A4}$$

Since, moreover, $u^i = c \cdot \delta^i_0$, (A3) can be transformed into

$$\Gamma^i_{0m} = \frac{1}{c^2}(w^i g_{0m} - \delta^i_0 w_m),$$

from which we obtain that the only nonzero components of the Christoffel-symbol are

$$\Gamma^{\mu}_{00}(\mathcal{L}) = \Gamma^{0}_{\mu 0}(\mathcal{L}) = \frac{1}{c^2} w^{\mu}. \tag{A5}$$

Notice that for a geodesic (A4) and (A5) reduces to

$$g_{ij}(\mathcal{G}) = \eta_{ij}, \qquad \Gamma^{i}_{jk}(\mathcal{G}) = 0.$$

For a general world line we have up to first order in x, y, z

$$g_{ij} = \eta_{ij} + a_{ij}x + b_{ij}y + c_{ij}z. \tag{A6}$$

If $w^i = constant$ the coefficients in (A6) are also constants.

Choose $k = 1$ ($x^1 \equiv x$) in the formula

$$\frac{\partial g_{ij}}{\partial x^k} - \Gamma^{l}_{ik} g_{lj} - \Gamma^{l}_{jk} g_{il} = 0$$

which expresses the vanishing of the covariant derivative of the metric tensor and substitute (A6) into it. We obtain

$$a_{ij} = \Gamma^{l}_{ix} \eta_{lj} + \Gamma^{l}_{jx} \eta_{il}.$$

Then from (A5) it follows that the only nonzero component of a_{ij} is $a_{00} = \frac{2}{c^2} w^x$. Analogous result may be obtained for b_{ij} and c_{ij} too, so we find that up to first order in $\vec{r} = (x, y, z)$

$$g_{ij} = \eta_{ij} + \frac{2}{c^2}(\vec{w} \cdot \vec{r}) \cdot \delta_{i0} \cdot \delta_{j0},$$

as indicated in Section 3.

* * *

Consider the quadratic form

$$ds^2 = F(z) \cdot c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

and try to choose $F(z)$ so as to make the Riemann-tensor to vanish.

The curvature 2-form $\mathcal{R}^i_{.j}$ has a single independent nonzero component, say

$$\mathcal{R}^z_{.0} = \left(\frac{F'^2}{4F} - \frac{1}{2}F''\right) d(ct) \wedge dz.$$

The function $F(z)$ must, therefore, satisfy the equation

$$F'' - \frac{F'^2}{2F} = 0,$$

the general solution of which is $F(z) = (a + bz)^2$. Hence

$$ds^2 = (a + bz)^2 c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{A7}$$

It is not difficult to prove that for an arbitrarily accelerating body of reference the generalization of (A7) is

$$ds^2 = \left(a + \vec{b} \cdot \vec{r}\right)^2 - dx^2 - dy^2 - dz^2$$

in which $a(t)$, $\vec{b}(t)$ is determined by fitting to the formula (1). When, moreover, the body of reference is rotating too the generalization of this formula is

$$ds^2 = \left[\left(a + \vec{b} \cdot \vec{r}\right)^2 - \frac{\omega^2 r^2}{c^2} - \frac{(\vec{\omega} \cdot \vec{r})^2}{c^2}\right] c^2 dt^2 -$$

$$-dx^2 - dy^2 - dz^2 + 2\,(\vec{\omega} \times \vec{r}) \cdot d\vec{r}\, dt.$$

In order to perform canonical quantization transformation from x, y, z, t to Minkowskian coordinates x', y', z', t' must be performed.

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