Desynchronization and Clock Paradox

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Abstract

A special relativistic phenomenon called desynchronization is discussed. Section 1 is devoted to heuristic explanation of the effect while a more formal derivation of it by means of an appropriate coordinate transformation is presented in the Appendix. The significance of desynchronization to clock paradox is clarified in Section 2.

1 Desynchronization

Consider a train at rest in the system of inertia $\mathcal{I}_0$ of the railway station, and let a pair of ideal clocks P and Q be fixed on it. Assume that they are synchronized correctly by light signals. Therefore, if a short light pulse is forced, by means of mirrors, to move to and fro between them, then the time $\overrightarrow{\Delta} t$ of flight of the signal from P to Q is precisely equal to the time $\overleftarrow{\Delta} t$ on the way back. If the distance between the clocks is l, then both time intervals are equal to l/c. In the example of Table 1 $\overrightarrow{\Delta} t = \overleftarrow{\Delta} t = 10$.

1.Moments of reflection by the clocks at rest

Moments of reflection at P	10		30		50		70
Moments of reflection at Q		20		40		60	

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Assume now that the train starts to move[1] in positive direction (to the right), P being behind Q. Accelerating gradually (adiabatically), it finally reaches velocity U and then continues to move uniformly with this velocity. Then the train will be a system of inertia again which will be denoted by $\mathcal{I}$. Due to the adiabaticity of acceleration the distance between the clocks on the uniformly moving train (in $\mathcal{I}$) remains equal to l.

During this process the light pulse continues to bounce between the clocks P and Q, but its times of flight towards and backwards are no longer equal to each other. As seen from $\mathcal{I}_0$(the embarkment) these time intervals satisfy the equations

$$c \cdot \overrightarrow{\Delta} t_0 = l_0 + U \cdot \overrightarrow{\Delta} t_0, \qquad c \cdot \overleftarrow{\Delta} t_0 = l_0 - U \cdot \overleftarrow{\Delta} t_0. \tag{1}$$

[1]In this paper only motions along straight line will be considered.

As indicated by index zero, in these equations the time intervals and the distance between the clocks are referred to $\mathcal{I}_0$. We will now express them through unindexed quantities valid in $\mathcal{I}$.

The geometrical meaning of the first of the above equations is that the light pulse reflected from P at some moment t_0 reaches Q, moving with the speed U, at the moment $t_0 + \overrightarrow{\Delta} t_0$ provided $\overrightarrow{\Delta} t_0$ is the solution of this equation. Since the clocks move with the speed U in $\mathcal{I}_0$, the proper time elapsed on both Q and P during this coordinate time interval is equal to $\overrightarrow{\Delta} t_0 \sqrt{1 - U^2/c^2}$:

$$\overrightarrow{\Delta} t = \overrightarrow{\Delta} t_0 \sqrt{1 - U^2/c^2}.$$

By analogous reasoning

$$\overleftarrow{\Delta} t = \overleftarrow{\Delta} t_0 \sqrt{1 - U^2/c^2}.$$

We note finally that l_0 is equal to the Lorentz-contracted distance between the clocks as seen in $\mathcal{I}_0$:

$$l_0 = l\sqrt{1 - U^2/c^2}.$$

Now, if in the solution

$$\overrightarrow{\Delta} t_0 = \frac{l_0}{c - U}, \qquad \overleftarrow{\Delta} t_0 = \frac{l_0}{c + U} \tag{2}$$

of (1) quantities related to $\mathcal{I}_0$ are expressed through their unindexed counterparts we arrive at the formulae

$$\overrightarrow{\Delta} t = \frac{l}{c}\left(1 + U/c\right), \qquad \overleftarrow{\Delta} t = \frac{l}{c}\left(1 - U/c\right) \tag{3}$$

for the time intervals of the rightward and leftward traveling of the pulse *as shown by the clocks P and Q themselves.* In the numerical example of Table 2 the former and the latter are equal to 12 and 8 respectively.

2.Moments of reflection by the moving clocks

Moments of reflection at P	10		30		50		70
Moments of reflection at Q		22		42		62	

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But measurement of light speed in the moving train would prove with certainty that it is equal to the same c in both positive and negative directions. Therefore, synchronicity of the clocks has been lost during acceleration. This phenomenon will be called *desynchronization.* It refers to *ideal clocks* which never break down and is *reversible* since synchronicity will be restored when the train stops again. Desynchronization is, therefore, a lasting *inertial effect* like that of the ball which starts moving backward when the train begins to accelerate and continues to roll uniformly when the train has reached its final velocity U. In neither case the effect is caused by any physical influence. On the contrary, it is the consequence of the inability of taking over the motion of the train: The ball remains at rest and the clocks remain synchronized with respect to the railway station (the system of inertia $\mathcal{I}_0$) even when the train is already moving. Saying it in a bit paradoxically, desynchronization happens because nothing happens to the mechanism of the clocks.

Were relativity theory wrong, desynchronization of ideal clocks would never occur. If ether existed and rested, say, with respect to the embarkment then Table 1 would retain its validity. Table 2 would also remain essentially true since, according to (2), $\overrightarrow{\Delta} t_0$ and $\overleftarrow{\Delta} t_0$ would be different from each other. But now this fact would be the consequence of the real difference of light speed in forward and backward direction rather than that of clocks' desynchronization.

Desynchronization in relativity theory is the direct consequence of the postulate that light velocity is the same in all inertial frames. As a consequence of this postulate, any system of inertia has its own set of correctly synchronized virtual clocks at rest which show Minkowskian coordinate time in it. Clocks P and Q may be considered as members of such a set $\mathcal{S}_0$, belonging to $\mathcal{I}_0$. Their desynchronization on the moving train is *an observable aspect*[2] of the fact, that they do not fit into the set $\mathcal{S}$ associated with $\mathcal{I}$.

As suggested by Table 2, the magnitude of desynchronization of Q and P is given by the formula

$$\Theta_l = \frac{1}{2}(\overrightarrow{\Delta} t - \overleftarrow{\Delta} t) = \frac{Ul}{c^2}. \tag{4}$$

In order to synchronize them either the hand of the clock Q, travelling ahead of P, must be moved back, or that of the clock P, travelling behind Q, must be moved forward by Θ_l. It is easy to see that (4) coincides with the magnitude of non-simultaneity in Einstein's train-platform thought experiment.

An important consequence of desynchronization can be elucidated if one assumes that a, say, 60 meter long car of the train is a physical laboratory in which a physicist is verifying the validity of the relativistic formula $d\tau = dt\sqrt{1 - v^2/c^2}$. He sets up an array of 60 ideal clocks along the car each of which is separated by one meter distance from its neighbours and carefully synchronizes them by means of light signals. They will serve to show Minkowskian coordinate time t while the proper time τ will be measured with physicist's ideal wristwatch.

Having finished synchronization, he begins, after some relaxation, the experiment by walking along the car with the velocity $v = 1\ m/s$, i.e. observing that each subsequent clock he passes by show precisely one second more than did the previous one. He finds that during the coordinate time interval $\Delta t = 60\ s$, required to finish his walk, the proper time $\Delta\tau$ elapsed on his wristwatch is less than 60 seconds, as he expected. But after having repeated the experiment several times he conludes that, contrary to the formula $d\tau = dt\sqrt{1 - v^2/c^2}$, two different values of $\Delta\tau$ are observed depending on the direction of his walk.

After some reflexion on the problem he remembers that synchronization was accomplished when the train was still staying in the railway station but when

[2] By means of suitable hardware the effect could probably be employed in a speedometer.

he began experimenting it was already travelling with some constant velocity. Meanwhile the clocks got desynchronized and this fact may account for the observed result.

Assume that the physicist moves in the same direction as the train does. Then every next clock is ahead the time it would show were it be synchronized correctly with the previous one. When, therefore, he reads 1 second more on the next clock as compared to the previous one, the proper time elapsed on his wristwatch will actually correspond to a Minkowskian coordinate time interval less than 1 second. On the way back just the opposite occurs. Therefore, according to his wristwatch, his walk in the direction of motion of the train should require less time than in the opposite direction with the same unit velocity:

$$\overrightarrow{\Delta}\tau < \overleftarrow{\Delta}\tau.$$

Generalization of the formula $d\tau = dt\sqrt{1 - v^2/c^2}$ to desynchronized coordinate time follows the same line of argument. Let us denote the distance between two adjacent clocks by dl and assume that, according to the desynchronized clocks, it takes a coordinate time interval dt to cover this distance. The speed is then equal to $v = \dfrac{dl}{dt}$. If, on the other hand, the two clocks were synchronized correctly then, according to (4), the corresponding time interval would be

$$d\bar{t} = dt - \frac{U dl}{c^2}, \tag{5}$$

leading to the speed $\bar{v} = \dfrac{dl}{d\bar{t}}$. Then in terms of barred quantites, the proper time interval elapsed on the path between the clocks is equal to $d\tau = d\bar{t}\sqrt{1 - \bar{v}^2/c^2}$. Since $\bar{v} = \dfrac{dt}{d\bar{t}}\dfrac{dl}{dt} = \dfrac{dt}{d\bar{t}}v$, this can also be written as

$$d\tau = dt \cdot \frac{d\bar{t}}{dt}\sqrt{1 - \left(\frac{dt}{d\bar{t}}\right)^2 v^2/c^2} = dt\sqrt{\left(\frac{d\bar{t}}{dt}\right)^2 - v^2/c^2}.$$

But, according to (5),

$$\frac{d\bar{t}}{dt} = 1 - \frac{Uv}{c^2},$$

and we obtain

$$d\tau = dt \times \sqrt{\left(1 - \frac{Uv}{c^2}\right)^2 - v^2/c^2}. \tag{6}$$

The velocity v is positive for walking along the motion of the train and negative in the opposite direction.

We see then that in desynchronized coordinates the relativistic space-time interval is given by the formula

$$\begin{aligned} ds^2 = c^2 d\tau^2 = \left(1 - \frac{Uv}{c^2}\right)^2 c^2 dt^2 - dl^2 = \\ = c^2 dt^2 - 2U\,dt\,dl - (1 - U^2/c^2)\,dl^2. \end{aligned} \tag{7}$$

Here and in what follows dl denotes displacement rather than distance.

2 Relation of desynchronization to clock paradox

If two objects meet each other twice than the proper time elapsed on one of them is in general different from that passed on the other. The proper times on the objects are shown by ideal clocks attached to them. No such phenomenon exists in Newtonian physics hence the name *clock paradox* (or *twin paradox*) has been given to it.

Theoretically the paradox is well understood. The proper time elapsed between two encounters is an invariant. Its value can be calculated in any coordinates once the trajectory is known in terms of them. In particular, when one of the objects is in the state of inertial motion then the longest proper time always belongs to it.

The only reason the paradox is worth of further discussion is that its illustration by the simplest conceivable thought experiment[3] seems to lead to the contradicting conclusion that the time passed on either of the two objects is shorter than the time passed on the other. The only purpose of the following discussion is to trace the error in this argument and correct it.

Consider the railway station at, say, **M** through which a train passes toward **N**. Arriving at **N** the train immediately reverses its course back to **M**. In **M** Alice observes transitions of the train in both directions and measures the time T_A between them. On the train Bob measures similarly the time interval T_B elapsed between their two encounters. What is the relative magnitude of the two time intervals?

According to relativity theory the answer is unambiguous: Since Alice stayed at rest in the system of inertia $\mathcal{I_A}$ of the station and Bob experienced acceleration at **N**, T_B will be shorter than T_A. If velocity reversal at **N** can be assumed instantaneous and the velocity of the train is equal to V in both directions then, due to time dilation in $\mathcal{I_A}$, $T_B = T_A\sqrt{1 - V^2/c^2} < T_A$.

If the distance between **M** and **N** is L then obviously

$$T_A = \frac{2L}{V} \qquad \text{and} \qquad T_B = \frac{2L}{V}\sqrt{1 - V^2/c^2}. \tag{8}$$

This conclusion, however, can be challenged by the following reasoning. Let us consider the situation from the rest frame $\mathcal{R_B}$ of Bob, with respect to which Alice at first moves away and then comes back again with the velocity V. Except for the instant of her velocity reversal, $\mathcal{R_B}$ is a system of inertia and as a result of time dilation her clock will go slower than Bob's. Then

[3] This thought experiment is discussed by Einstein in his *Dialog* [1] where he suggests an explanation in the framework of general relativity (see also [2]). Below it will be shown that full clarification of the apparent contradiction is provided already by special relativity.

$T_A = T_B\sqrt{1 - V^2/c^2} < T_B$ which is just the opposite of the previous — undisputably correct — conclusion. It seems that the only remedy might be a jump in the pointer position of one, or both, of the clocks at the moment of velocity reversal but, since no such leap was assumed to happen in the discussion from the point of view of Alices's rest frame, such a loophole is closed.

Nevertheless, this argumentation is certainly flawed. In order to cure it we first reconsider the correct consideration with respect to $\mathcal{I}_\mathcal{A}$ in more detail and then apply *exactly the same procedure* to the alternative description from the point of view of the train (i.e. in $\mathcal{R}_\mathcal{B}$).

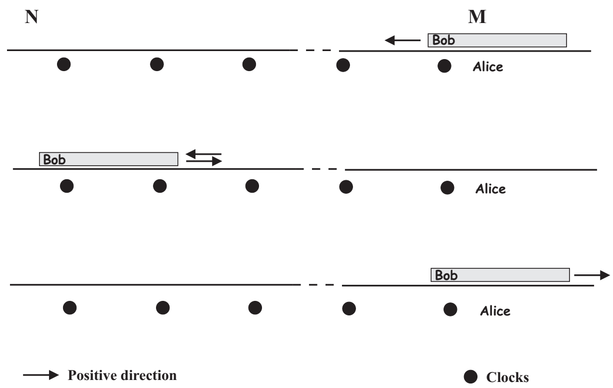

Figure 1

At first sight it seems that, working in Alice's rest frame, the formula $d\tau = dt\sqrt{1 - V^2/c^2}$ is applicable authomatically since $\mathcal{I}_\mathcal{A}$ is an inertial frame. But this is not quite true. Quantity t is the ingredient of the space-time coordinate system and relations like $d\tau = dt\sqrt{1 - V^2/c^2}$, containing it, can be formulated only if it has already be unambiguously defined. It is known (and has been also demonstrated in the previous section) that this formula is valid only in Minkowskian (pseudo-orthogonal) coordinate system. According to special relativity such a coordinate system is always available but it can serve as a *rest frame* only for a body performing inertial motion. Since for Alice this condition is fulfilled we choose a Minkowskian space-time coordinate system in which the whole environment containing both **M** and **N** is at rest.

Coordinate time t_A is then shown by clocks synchronized in $\mathcal{I}_\mathcal{A}$ by light signals and densily distributed in it. In general they are virtual but some of them may be realized as shown on Fig. 1. On his way from **M** to **N** and back Bob may record time τ_B elapsed on his own clock as a function of the coordinate time t_A read off on the clocks he passes by. This function is shown on Fig. 2a. Its slope is equal to $\sqrt{1 - V^2/c^2}$.

Now an exactly analogous protocol must be applied also in the rest frame $\mathcal{R}_\mathcal{B}$ of Bob. At first a space-time coordinate system must be selected in which his train is at rest. Now it cannot be Minkowskian since Bob's train does not

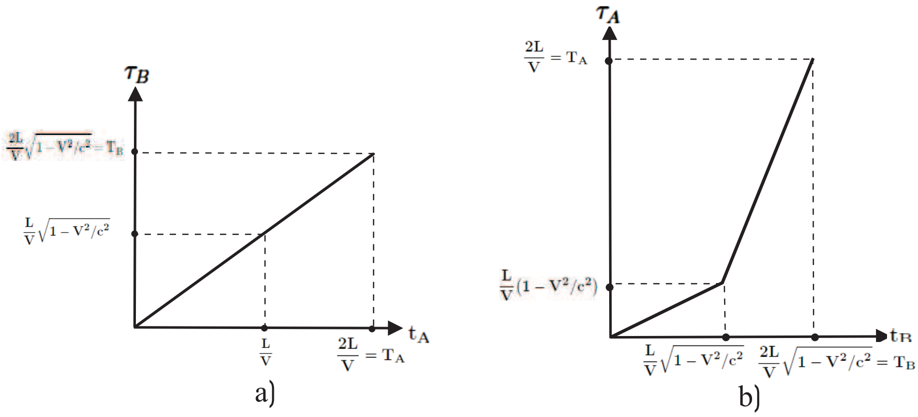

Figure 2

perform inertial motion. But we may assume that at least on the way from **M** to **N** the virtual clocks on the train which show the coordinate time t_B are synchronized there by light signals. Some of these clocks may be realized again as shown on Fig. 3. As seen on the figure the train is depicted long enough to capture both stations but in an idealized thought experiment such an exaggeration is obviously permitted.

On her way toward the rear end of the train and backward Alice may also record the proper time τ_A shown by her clock as a function of t_B read off on the clocks she passes by. But now the slope of this function will be equal to $\sqrt{1 - V^2/c^2}$ only in the first part of her way since after reversal the coordinate time t_B becomes desynchronized and, according to (6), the slope of the function $\tau_A(t_B)$ will be equal to $\sqrt{\left(1 - Uv/c^2\right)^2 - v^2/c^2}$ rather than $\sqrt{1 - V^2/c^2}$.

In order to calculate this slope we should express velocities U and v as functions of V. The first of them is, by definition, equal to train's velocity gain during velocity reversal. According to relativistic velocity addition

$$U = \frac{2V}{1 + V^2/c^2} \tag{9}$$

In order to calculate Alice's velocity v after reversal let us temporarily omit index B from t_B and introduce again $\bar{t}$ as in (5). Then

$$\frac{dl}{dt} = v, \qquad \text{and} \qquad \frac{dl}{d\bar{t}} = -V.$$

To express the former in terms of the latter we write

$$v = \frac{dl}{dt} = \frac{dl}{d\bar{t}} : \frac{dt}{d\bar{t}} = -V : \frac{dt}{d\bar{t}}.$$

For the last term (5) gives

$$\frac{dt}{d\bar{t}} = 1 + \frac{U}{c^2}\frac{dl}{d\bar{t}} = 1 - UV/c^2.$$

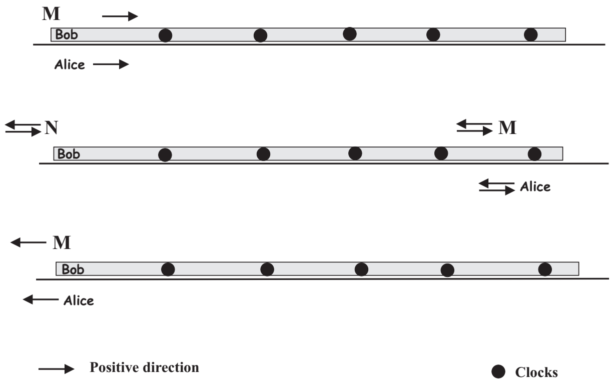

Figure 3

Hence

$$v = -V \frac{1}{1 - UV/c^2}.$$

Substituting U from (9) we finally obtain

$$v = -V \frac{1 + V^2/c^2}{1 - V^2/c^2}. \tag{10}$$

Using these equations we obtain for the slope of the function $\tau_A(t_B)$ after velocity reversal the expression

$$\sqrt{\left(1 - \frac{Uv}{c^2}\right)^2 - v^2/c^2} = \frac{1 + V^2/c^2}{\sqrt{1 - V^2/c^2}}. \tag{11}$$

We have thereby found the error in the naiv treatment of our thought experiment: It consisted in the use of the slope $\dfrac{d\tau_A}{dt_B} = \sqrt{1 - V^2/c^2}$ for Alice's clock on her way back which contradicts the behaviour of the ideal clocks showing coordinate time in the rest frame of Bob. At the same time the correct slope (11) verifies (8) obtained in a much simpler way in the rest frame of Alice.

With respect to the train the distance between **M** and **N** is $L\sqrt{1 - V^2/c^2}$. In the firs part of her motion Alice covers this distance over the coordinate time interval $\overrightarrow{\Delta} t_B = \dfrac{L\sqrt{1 - V^2/c^2}}{V}$. Her way back requires the same time since — as we have emphasized in the previous section — desynchronization reflects the inability of the clocks to adapt to the new speed of the train. Hence

$$\overrightarrow{\Delta} t_B = \overleftarrow{\Delta} t_B = \frac{L\sqrt{1 - V^2/c^2}}{V}.$$

Then the proper time interval T_A is easily obtained:

$$T_A = \frac{L\sqrt{1-V^2/c^2}}{V}\left\{\sqrt{1-V^2/c^2}+\sqrt{\left(1-\frac{Uv}{c^2}\right)^2-v^2/c^2}\right\} =$$
$$= \frac{L}{V}\left\{(1-V^2/c^2)+(1+V^2/c^2)\right\} = \frac{2L}{V},$$

in conformity with (8).

The value of T_B follows from the observation that since Bob is at rest on the train his clock runs in the rythm of the coordinate time t_B. Hence

$$T_B = \overrightarrow{\Delta} t_B + \overleftarrow{\Delta} t_B = \frac{2L\sqrt{1-V^2/c^2}}{V}$$

as expected. The function $\tau_A(t_B)$ is shown on Fig. 2b.

In order to have our task completely finished we should notice that the original formulation of clock paradox refers to a pair of naked clocks A and B without any auxiliary device like the train and arrays of clocks. In a *measurement* of T_A and T_B coordinates play no role at all and functions $\tau(t)$ are even not defined. But *calculation* of T_A and T_B is possible only in some specified coordinate system. Then a function $\tau(t)$ arises also automatically as by-product.

Since T_A and T_B are invariants, the coordinate system may be chosen arbitrarily. Simplicity favors, of course, Minkowski-coordinates. But, as we have seen, sometimes coordinates are required to be attached to one of the clocks. If it is the accelerating one than the coordinate system will not be pseudo orthogonal and virtual clocks, showing coordinate time, will be in general desynchronized[4]. The tale of Bob's trip may help visualizing this.

Appendix

In this Appendix desynchronization will be analysed in terms of coordinate transformations. A coordinate system will be called *attached* to a reference frame (body) if both the body's parts and the (virtual) clocks, showing coordinate time, are at rest in it and, in addition, any external manipulation of the clocks (as e.g. their resyncronization) is excluded. For example, the coordinates (t_0, x_0) in Section 1 are attached to the railway embankment and also to the train while it is at rest in the station. When the train is moving with constant speed the coordinates

$$\bar{t} = \frac{t_0 - \frac{U}{c^2}x_0}{\sqrt{1-U^2/c^2}} \qquad \bar{x} = \frac{x_0 - Ut_0}{\sqrt{1-U^2/c^2}} \tag{A.1}$$

are attached to it.

[4]The relationship of clock paradox to desynchronization has already been pointed out in [3].

In order to attach coordinates to the train over its whole history we split the Lorentz-transformation (A.1) into two steps:

$$(t_0, x_0) \longrightarrow (t, x) \longrightarrow (\bar{t}, \bar{x}).$$

The individual steps are given by the formulae

$$t = t_0\sqrt{1 - U^2/c^2} \qquad x = \frac{x_0 - Ut_0}{\sqrt{1 - U^2/c^2}}, \tag{A.2}$$

and

$$\bar{t} = t - \frac{U}{c^2}x \qquad \bar{x} = x. \tag{A.3}$$

All these coordinates occurred in Section 1 (through their time components). The coordinate time t is shown by clocks on the moving train. In (A.2) this is ensured by the identification of t with the proper time of clocks even after the train has started to move in $\mathcal{I}_0$. Coordinate time t is desynchronized since these clocks were synchronized when the train was still standing in the station. Coordinates $(\bar{t}, \bar{x})$ are obtained by subsequent resynchronization of t on the moving train.

To demonstrate explicitely the correctness of this identification let us calculate the spacetime interval in terms of (t, x):

$$ds^2 = c^2 dt_0{}^2 - dx_0{}^2 = c^2 dt^2 - 2U dt\, dx - \left(1 - U^2/c^2\right) dx^2. \tag{A4}$$

It coincides with (7) as expected.

Light velocities

$$c_\pm = \left(\frac{dx}{dt}\right)_\pm$$

in the positive and negative directions are obtained from $ds^2 = 0$ upon dividing it by dt^2:

$$\left(c - \frac{U}{c}c_\pm\right)^2 - c_\pm^2 = 0.$$

Solutions of this equation are

$$c_\pm = \pm\frac{c}{1 \pm U/c}.$$

Hence, the propagation times are

$$\overrightarrow{\Delta} t = \frac{l}{c_+} = \frac{l}{c}\left(1 + U/c\right), \qquad \overleftarrow{\Delta} t = \frac{l}{|c_-|} = \frac{l}{c}\left(1 - U/c\right)$$

in accordance with (3).

A coordinate system, say (θ, ξ), attached to the train throughout its whole history can now be easily defined. Assuming that acceleration is instantaneous and takes place at $t_0 = t = 0$, we may put

$$(\theta, \xi) = \begin{cases} (t_0, x_0) & \text{if } t_0 < 0, \\ (t, x) & \text{if } t > 0. \end{cases} \tag{A.4}$$

This transformation is at $t_0 = t = 0$ discontinuous (except for the origin). Hence wordlines $\xi = const$ of objects fixed on the train have, at the moment of acceleration, a discontinuity *in spacetime*, reflecting the deformation of the train in the limit of instantaneous acceleration. On Fig. 4 the spacetime domain occupied by the train is seen together with the world lines $\xi = const$ of some objects fixed on it.

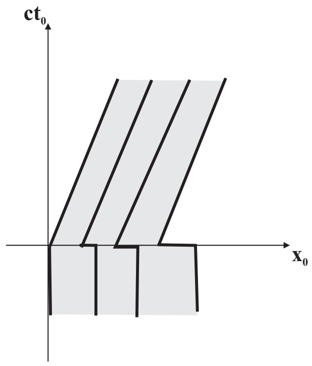

Figure 4

Now both the coordinates (t_0, x_0) and $(\bar{t}, \bar{x})$ are pseudo orthogonal and if an object at rest in either of them occupies the interval (a, b) than its length is equal to $b - a$. Owing to (A.3), the same is true for (t, x). Therefore, the distance between fixed objects remains the same both before and after acceleration but it becomes contracted in $\mathcal{I}_0$ when the train is moving. Obviously the same is true for the train itself.

Therefore, our description of the train is in full agreement with the absence of rigid bodies in relativity. Moreover, though (A.4) describes instantaneous acceleration, its consequences coincide with those of an adiabatic one.

World lines of objects not attached to the train are, of course, continuous.

In Section 2 the coordinate system (t_0, x_0) played the role of coordinates attached to the incoming train. Then (t_B, x_B) attached to Bob both before and after velocity reversal is identical to (θ, ξ). The trajectory of Alice is continuous in it and the same is true for Bob if he is sitting in the origin. This coordinate description would permit us to draw conclusions of Section 2 by almost mechanical calculations. There is no need to actually do that but to give an example we rederive (10).

Alice's trajectory in the coordinates attached to Bob at $t_0 \leq 0$ is $x_0 = Vt_0 + L\sqrt{1 - V^2/c^2}$. At $t \geq 0$ this equation must be expressed in terms of (t, x). Using (A.2) we obtain

$$x\sqrt{1 - U^2/c^2} + \frac{U}{\sqrt{1 - U^2/c^2}}t = V\frac{t}{\sqrt{1 - U^2/c^2}} + L\sqrt{1 - V^2/c^2}.$$

We now differentiate this equation and solve it with respect to $\dot{x}$:

$$v = \frac{dx}{dt} = -\frac{U - V}{1 - U^2/c^2}.$$

Substituting here $U = 2V/(1 + V^2/c^2)$ we arrive at (10) again.

References

[1] A. Einstein, *Eine Dialog über Einwände gegen die Relativitätstheorie*, Naturwiss., **6**, 697–702, 1918.

[2] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, 1969).

[3] P. Hraskó, *Basic Relativity — An Introductory Essay* (Springer, 2012).