Theoretical Aspects of the GP-B Experiment

Based on the lecture given at the Summer School on Gravity Theory
Gyöngyöstarján, Hungary, 2006

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Contents

Appendices

The aim of the GP-B experiment was to study geodetic precession and drag. Below these phenomena will be discussed from various points of view.

1 The concept of the rotation-free motion in curved space-time.

The absence of rotation of moving bodies is testified by the direction of the spin of (miniature) spherical *test gyroscopes* carried by them[1]. Therefore, the term 'rotation' applied to a test gyroscope disregards rotation of its rotor and refers exclusively to the change of orientation of its *axis of rotation*. It then follows that the physical criterion of rotationlessness of a *spacecraft* is that the axes of test gyroscopes point continuously at the same piece of the capsule's wall. According to this definition both the free and accelerating motions of a spacecraft may be either rotating or rotation-free.

In order to formulate the *mathematical criterion* of the absence of rotation let us imagine a spacecraft which moves freely without rotation along the geodetic $\mathcal{G}$ (the rotationlessness of accelerating spacecrafts will be considered in Chap. 4). As a consequence of equality of the inertial and gravitating mass (weak principle of equivalence) our spacecraft is a local inertial frame in the sense that bodies at rest in it left to themselves remain motionless (weightlessness). However, according to the strong principle of equivalence, the spacecraft must be a true inertial frame in every other respect too because special relativity is valid in it. In particular, gyroscopes must conserve the direction of their axes relative to it. Geodetic hypotheses is, however, insufficient to ensure this because it is consistent with the rotation of the spacecraft moving along geodetic. It must be, therefore, clarified whether the structure of the pseudoriemannian spacetime permits rotation free geodetic motion and if so how to formulate the condition of absence of rotation covariantly.

The mathematical criterion of the rotation-free motion of local (but not strictly pointlike) frames of reference can be arrived at in four steps:

1. The mathematical condition of the strong equivalence principle should be formulated.

2. It must be demonstrated that this condition is fulfilled on the timelike geodetics of the pseudo-riemannian spacetime.

3. The equation determining the *spatial orientation* of spacecrafts should be deduced from the requirement that they be local inertial frames.

[1]The spherical shape of the gyros serves to exclude any torque acting on them. The Earth itself for example in not from this point of view a good test gyroscope on the scale of the Solar System because of the quadrupole moment due to its Equatorial bulge. This quadrupole moment leads to the nutation of the axis of rotation and to the lunisolar precession of 25700 years period. These peculiar motions forbid us to assume that the Earth's axis of rotation moves 'parallel to itself' and so the Earth *does not* perform rotationless motion around the Sun.

4. The covariant form of the equation determining the direction of the spin of a gyroscope must be found.

Let us now discuss these points in some more detail.

1. Let the equation of the geodetic $\mathcal{G}$ be $x^i = x^i(\tau)$. If in the region of spacetime occupied by the spacecraft special relativity was strictly true then it could be covered by Minkowski-coordinates for which $g_{ij} = \eta_{ij}$ and $\Gamma^i_{jk} = 0$. But in this 'tube' spacetime is in general curved and Minkowski coordinates can exist in it at most *locally* and the above conditions should be fulfilled only along $\mathcal{G}$.

   The criterion for the validity of the equivalence principle can, therefore, be summarized in the following requirement: *To every timelike $\mathcal{G}$ there should exist a coordinate system in which we have on $\mathcal{G}$*

$$g_{IJ}(\mathcal{G}) = \eta_{IJ}, \qquad \Gamma^I_{JK}(\mathcal{G}) = 0 \tag{1.1}$$

   and the coordinate time is identical (up to an irrelevant constant) *to the proper time of $\mathcal{G}$: $dt = d\tau$*. This coordinate system if exists is called *Fermi coordinate system*[2].

   The time axis of the Fermi coordinates is identified with the geodetic $\mathcal{G}$. The identity of the Fermi coordinate time with the proper time along $\mathcal{G}$ is the consequence of the fact that the clocks the cosmonauts use in their experiments show proper time[3]. The vanishing of the connection coefficients (i.e. of partial derivatives of the metric tensor) on $\mathcal{G}$ makes the equivalence principle a physically meaningful concept for reference frames of small but finite extension.

2. If in (1.1) one replaces $\mathcal{G}$ by an arbitrary point $\mathcal{P}$ then, according to a well known theorem of Riemannian geometry, this pair of conditions can always be fulfilled by an appropriate choice of the coordinate system. In (pseudo-)riemannian geometry even this weaker condition requires careful proof and it is the more so when the theorem is extended to geodetics.

   It can be shown[4] that the generalization of this theorem to timelike geodetics is indeed true and so a spacecraft if not extremely large[5] can indeed be a (local) inertial frame in which special relativity is valid. To this end it must be at rest in the Fermi coordinate system.

3. The coordinate basis of the Fermi coordinate system on $\mathcal{G}$ will be denoted by $\mathbf{e}_{(I)}(\tau)$. Its timelike element is proportional to the four-velocity of $\mathcal{G}$:

[2]Latin and greek indices take on values $0, 1, 2, 3$ and $1, 2, 3$ respectively. Components in the Fermi coordinate system are distinguished by upper case indices.

[3]For example, according to the equivalence principle light velocity in the spacecraft is isotrope. Verifying this by e.g. in a series of rotating mirror experiments of Fizeau cosmonauts measure both path and time using their own instruments at rest in the spacecraft.

[4]See Appendix 2.

[5]It is the magnitude of the tidal forces by which it can be decided whether a spacecraft of given size can be considered an inertial frame from the point of view of a given problem.

$\mathbf{e}_{(O)}(\tau) = \mathbf{V}(\tau)/c$: The Fermi coordinate base on $\mathcal{G}$ is *line oriented*. Line orientation ensures that the Fermi coordinate time t on $\mathcal{G}$ is identical to the proper time of $\mathcal{G}$ since in this case $e^O_{(O)} = V^O/c = \dfrac{dt}{d\tau}$ while $e^O_{(O)} = 1$ by definition.

The spacelike elements $\mathbf{e}_{(\Lambda)}(\tau)$ can be chosen arbitrarily only at a single point of $\mathcal{G}$ (at say $\tau = 0$) since, as it turns out in the proof (in Appendix 2) of the existence of Fermi coordinates, the elements of this basis must satisfy on $\mathcal{G}$ the *equation of parallel transport*

$$\frac{De^i_{(\Lambda)}}{d\tau} = \frac{de^i_{(\Lambda)}}{d\tau} + \Gamma^i_{jk} V^j e^k_{(\Lambda)} = 0 \qquad \left(V^i = \frac{dx^i}{d\tau}\right) \tag{1.2}$$

(their absolute derivatives are zero). The name expresses the fact that solution of this equation is a vector field on $\mathcal{G}$ each element of which can be obtained from any other by parallel transportation. Eq. (1.2) respects line orientation since on $\mathcal{G}$ we have $\dfrac{D\mathbf{V}}{d\tau} = 0$.

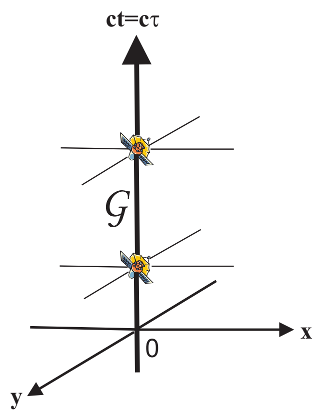

Figure 1

A spacecraft moving on $\mathcal{G}$ is a local inertial frame if its orientation is determined by the basis $\mathbf{e}_{(\Lambda)}(\tau)$. On Fig.1 the position of the spacecraft is shown in the Fermi coordinate system in two consecutive moments of time. The exact equality of the metric tensor to η_{ij} takes place only on the time axis but, owing to the vanishing of the connection coefficients on $\mathcal{G}$, this may be a good approximation in a sufficiently small tube around it.

4. Let us write down finally the equation satisfied by the direction of the gyroscope axis (by its spin). This spin is a three-vector $\vec{S}$ which is constant in the Fermi coordinates. In other words it is a linear combination of the elements $\mathbf{e}_{(\Lambda)}(\tau)$ with coefficients independent of τ (which is the Fermi

coordinate time on $\mathcal{G}$). $\vec{S}$ can be extended to a four vector $\mathbf{S}$ by requiring its time component to vanish in Fermi coordinates. This amounts to subject it to the covariant condition $\mathbf{S}(\tau) \cdot \mathbf{V}(\tau) = 0$.

Since the equation (1.2) of parallel transport is linear the components of $\mathbf{S}$ also satisfy it:

$$\frac{DS^i}{d\tau} = \frac{dS^i}{d\tau} + \Gamma^i_{jk} V^j S^k = 0. \tag{1.3}$$

This is the covariant equation of the rotationless geodetic motion of a test gyroscope. The meaning of Fermi coordinates ensures that in these coordinates on $\mathcal{G}$ we have $\frac{D}{d\tau} = \frac{d}{dt}$.

2 Rotation-free revolution in Newtonian physics.

Consider a spacecraft which freely revolves around the Earth on a circular orbit. While orbiting it may or may not rotate. How can the two cases be distinguished from each other visually?

Assume for simplicity that the the Earth neither rotates nor revolves around Sun and so the gyroscopes in laboratories on the ground point constantly to the same piece of the wall. Since this 'idealized Earth' is an inertial frame equations of motion in coordinate systems $\mathcal{K}$ attached to it do not contain inertial forces. But according to Newtonian physics gravity is a true force and the Newtonian equation of motion describing the motion of the spacecraft will contain the force of gravitation on its right hand side.

In what follows (until Chapt.6) we confine ourselves to rotation of the spacecraft in the plane of its revolution. Then it is natural to assume that test gyroscopes carried by it have their axes to lie in this plane too. In general the coordinate system $\mathcal{K}$' attached to the spacecraft rotates with the angular velocity w with respect to $\mathcal{K}$. The Coriolis force due to this rotation exerts a torque on the gyroscopes fixed in the spacecraft which keeps their orientation intact with respect to $\mathcal{K}$ (see Appendix 1 for the proof of this.). According to the physical criterion of the absence of rotation formulated at the beginning of the previous chapter the condition of the absence of rotation is, therefore, the vanishing of w: $w = 0$. But the spacecraft is not an inertial frame even in this case because the centripetal acceleration of its origin is equal to $a = v^2/r$ which in the Newtonian equation of motion written down in $\mathcal{K}$' is represented by the outward directed radial inertial force $ma = mv^2/r$ (this in not a centrifugal force because now $\mathcal{K}$' is rotation-free).

Observing the spacecraft from the 'idealized Earth' it is easy to establish whether it is rotating or not because as we have just seen its rotation is to be related to the Earth (to the coordinate system attached to Earth). Assume that the spacecraft is of the form of an elongated cigar. If we observe its consecutive transitions above a point on the Earth in its plane of revolution and find that its dimension is unchanged then the revolution is rotationless[6]. If on the contrary

[6] This is not quite true because the same is observed if w is an integer multiple of the

it does rotate then the angular velocity w of its rotation can be calculated by the rate of change of its observed dimensions. In this procedure based on the angle of sight we observe essentially the angle between the longitudinal axis and the perpendicular to the Earth's surface at the point of observation.

Though this Newtonian situation is simple enough extension to general relativity in the next chapter requires some further considerations. If polar coordinates are imagined around the Earth whose local coordinate base in the plane of revolution consists of the ordered pair of vectors $\mathbf{e}_{(r)}$ and $\mathbf{e}_{(\varphi)}$ pointing radially outward and in the direcion of revolution respectively then we see that the spacecraft rotates with respect to this local base with some angular velocity. When the motion is rotationless or the rotation is sufficiently slow then this angular velocity is negative and hence will be denoted by $-\Omega$. On the other hand the angular velocity of revolution denoted by ω is positive by definition (i.e. the direction of revolution will be chosen as the positive direction).

It is easy to see that the angular velocity w of rotation of the spacecraft is equal to the difference

$$w = (-\Omega) + \omega = \omega - \Omega. \tag{2.1}$$

Indeed, the angular velocity of the rotation with respect to the local coordinate basis is by definition equal to $-\Omega$ while the local coordinate basis itself rotates 'under the spacecraft' with the angular velocity ω. w is the resultant of these two angular velocities. For a rotationless motion the two constituents just compensate each other: The spacecraft rotates backward with respect to the local base with the same speed this local base rotates 'under it' in positive direction with respect to the, say, Cartesian coordinates attached to the Earth which are, contrary to the polar coordinates, regular in the origin[7].

3 Rotation-free revolution in general relativity.

In general relativity the problem treated in the preceding chapter under Newtonian paradigm is handled quite differently. In the Newtonian treatment we assumed the existence of the global *inertial reference frame* determined by the 'idealized Earth' with the *coordinate system* $\mathcal{K}$ attached to it. In general relativity there exist only local inertial frames realized by spacecrafts performing geodetic rotationless motion. Therefore, any reference to a global inertial reference frame is out of question. The calculation of the precession of gyroscopes carried by freely orbiting spacecrafts require[8]

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angular velocity of revolution. But in this introductory paragraph we may assume that this is not the case.

[7]Moon turns always the same side toward Earth. This means that in the local coordinate base her orientation is not changing ($\Omega = 0$) and so $w = \omega$. In other words the speed of its rotation is equal to the speed of its revolution around the Earth.

[8]All these steps have their counterparts in the Newtonian theory Einstein-equation being replaced, of course, by $\triangle\Phi = 4\pi G\rho$. However, the absence of any inertial force on the r. h. s. of the planets' equations of motion reveals the tacit assumption that the coordinate system chosen is attached to a global inertial reference frame.

1. Choice of a coordinate system,
2. solution (with appropriate boundary conditions) of the Einstein-equation around the Earth,
3. solution of the geodetic equation in this spacetime and
4. solution of the equation of parallel transport along the geodetics.

But *from the physical point of view* the absence of rotation means the same in both Newtonian and general relativistic physics. As we have already seen in Chapt.1 in this latter case it leads to the following condition: The spin four vectors **S** of test gyroscopes in the spacecraft must obey the equation of parallel transport (1.3) along the geodetic of the spacecraft. For a circular geodetic the angular velocity ω of revolution and the radius r of the trajectory are connected by the same equation

$$\omega = \sqrt{\frac{MG}{r^3}} = c\sqrt{\frac{r_g}{2r^3}}, \qquad r_g = \frac{2MG}{c^2}. \tag{3.1}$$

in both the newtonian and the relativistic case[9]. Since the solutions U^i of (1.3) are components in the local coordinate base their knowledge permits one to deduce the value of the angular velocity Ω for which the expression

$$\Omega = \omega\sqrt{1 - \frac{3}{2}\frac{r_g}{r}} \tag{3.2}$$

is obtained[10] (the quantity Ω has been introduced for just this reason).

Though this solution refers to the *rotationless* revolution it leads through (2.1) to a nonzero value for w:

$$w = \omega - \Omega = \omega\left(1 - \sqrt{1 - \frac{3}{2}\frac{r_g}{r}}\right) > 0. \tag{3.3}$$

If, therefore, we observe visually a spacecraft which performs rotation-free revolution (in the sense of Chapt.1) then we find that its orientation changes continuously i.e. it precesses with the angular speed w in positive sense. This phenomenon is called *geodetic precession* (A. D. Fokker 1920). In Newtonian physics $w = 0$ and no geodetic precession exists.

Geodetic precession is an observable phenomenon only because the element $\mathbf{e}_{(r)}$ of the coordinate base to which it is related can be defined operationally (without reference to any coordinates) by the position of the central body. When no such preferred direction exists no geodetic precession can be meaningfully defined.

[9] See Appendix 5. This coincidence, however, is of no intrinsic geometrical significance because in Newtonian physics r is the distance of the spacecraft from the center of the central body while in general relativity it is the radial coordinate which can be freely rescaled.

[10] Formula (3.2) is a special case of the more general relation (4.6) proved in Appendix 5.

Classical tests of general relativity theory concern Einstein-equation and geodetic hypothesis. Geodetic precession is sensitive to the equation of parallel transport too which is a further element of the theory. But to appraise its full significance we must pass beyond this mere technical aspect of this phenomenon.

Consider e.g. the anomalous precession of the perihelion of Mercury. Already astronomers of the middle of 19. century knew that the perihelion precession of the Mercury is ahead of the value calculated from Newtonian theory by about 40 arcsec/century. As a very first application of his theory Einstein was able to show that general relativity explains this anomaly to a high degree of precision. The significance of this agreement is difficult to overestimate but this result in itself says nothing of the outstanding originality of the new theory. It left open the possibility that the anomaly could be understood within the Newtonian framework too if all the relevant factors were taken into consideration some of which might have remained for those times hidden.

That was such a realistic possibility that it came almost true. In 1966 R. H. Dicke and H. M. Goldenberg performed new measurement of the equatorial bulge of the Sun and came to the conclusion that its improved value adds 3.4 arcsec/century to the previous Newtonian value of the perihelion precession which if were true would significantly spoil the agreement of the general relativistic value with the observation.

In contrast to perihelion precession geodetic precession is a phenomenon not foreseen in Newtonian gravity at all. Imagine that several spacecrafts were revolving around the Earth all without rotation (in the sense of Chapt. 1). If visual observation indicated at least qualitatively that they all *rotate* with different angular velocities that this fact alone would raise the most serious doubts on the very foundations of the Newtonian physics.

4 The rotationless accelerating motion.

Let the spaceraft move on wordline $\mathcal{L}$ which is not a geodetic because its four-acceleration is different from zero:

$$\mathbf{A} \overset{def}{=} \frac{D\mathbf{V}}{d\tau} \neq 0. \tag{4.1}$$

In this case condition (1.1) cannot be satisfied. This mathematical fact agrees with the equivalence principle according to which accelerating reference frames are not inertial frames. But the absence of rotation is ensured already by the weaker condition

$$g_{IJ}(\mathcal{L}) = \eta_{IJ}, \qquad \Gamma^{\Lambda}_{O\Sigma}(\mathcal{L}) = 0 \tag{4.2}$$

together with the requirement of line orientedness (O is the time index while Λ and Σ are spacelike indices). The reason for this is that it is the $\Gamma^{\Lambda}_{O\Sigma}$ type connection coefficients which describe the rotation of the pseudo-orthogonal basis during parallel transport and lead to the appearance of Coriolis force in rotating spacecrafts (see the end of Appendix 2). This last circumstance is important because, as shown in Appendix 1, deflection of the gyroscope axis is caused by the Coriolis force.

Assume that the spacelike elements of the Fermi coordinate basis $\mathbf{e}_{(\Lambda)}(0)$ on $\mathcal{L}$ are chosen at the point $\tau = 0$ of $\mathcal{L}$. Then it can be proved[11] that the basis elements $\mathbf{e}_{(\Lambda)}(\tau)$ are again uniquely determined along $\mathcal{L}$, this time by the conditions (4.2), since they should obey an equation analogous to (1.2) in which the absolute derivative is replaced by the *Fermi-Walker derivative*:

$$\frac{{}^*D\mathbf{e}_{(\Lambda)}}{d\tau} \stackrel{def}{=} \frac{D\mathbf{e}_{(\Lambda)}}{d\tau} - \frac{1}{c^2}\left[(\mathbf{V}\cdot\mathbf{e}_{(\Lambda)})\mathbf{A} - (\mathbf{A}\cdot\mathbf{e}_{(\Lambda)})\mathbf{V}\right] = 0.$$

Rotationless motion of any four-vector $\mathbf{U}$ satisfies this same equation:

$$\frac{{}^*D\mathbf{U}}{d\tau} = \frac{D\mathbf{U}}{d\tau} - \frac{1}{c^2}\left[(\mathbf{V}\cdot\mathbf{U})\mathbf{A} - (\mathbf{A}\cdot\mathbf{U})\mathbf{V}\right] = 0. \qquad (4.3)$$

When $\mathbf{A} = 0$ (4.3) reduces to the equation of parallel transport. Since as a consequence of (4.1), $\mathbf{V}\cdot\mathbf{A} = 0$ and $\mathbf{V}^2 = c^2$ we have $\frac{{}^*D\mathbf{V}}{d\tau} = 0$ and so equation (4.3) is consistent with the line orientedness of the Fermi coordinate base. The meaning of Fermi coordinates ensures that in these coordinates we have on $\mathcal{L}$ $\frac{{}^*D}{d\tau} = \frac{d}{dt}$.

When $\mathbf{U}$ is the spin $\mathbf{S}$ of a test gyroscope (4.3) can be simplified. In the rest frame of the gyro $\mathbf{S} = (\vec{S},\, 0)$ and, therefore, the last term in (4.3) vanishes: $\mathbf{V}\cdot\mathbf{S} = 0$:

$$\frac{D\mathbf{S}}{d\tau} + \frac{1}{c^2}(\mathbf{A}\cdot\mathbf{S})\mathbf{V} = 0.$$

From this equation[12] acceleration can be eliminated using orthogonality of $\mathbf{S}$ and $\mathbf{V}$:

$$\mathbf{S}\cdot\mathbf{A} = \mathbf{S}\cdot\frac{D\mathbf{V}}{d\tau} = \frac{D}{d\tau}(\mathbf{S}\cdot\mathbf{V}) - \mathbf{V}\cdot\frac{D\mathbf{S}}{d\tau} = -\mathbf{V}\cdot\frac{D\mathbf{S}}{d\tau}.$$

The transformed equation in component form can be written as

$$\frac{DS^i}{d\tau} - \frac{1}{c^2}V_j\frac{DS^j}{d\tau}V^i = \left(\delta^i_j - \frac{1}{c^2}V^iV_j\right)\frac{DS^j}{d\tau} \equiv \left(\frac{D\mathbf{S}}{d\tau}\right)^i_\perp = 0. \qquad (4.4)$$

The index $\perp$ in $\left(\frac{D\mathbf{S}}{d\tau}\right)_\perp$ refers to the transverse component with respect to $\mathbf{V}$, and so the number of independent component equations is three. These must be implemented by the condition of orthogonality of $\mathbf{S}$ and $\mathbf{V}$. Therefore, the complete system of equations for $\mathbf{S}$ is

$$\left(\frac{D\mathbf{S}}{d\tau}\right)_\perp = 0, \qquad \mathbf{S}\cdot\mathbf{V} = 0. \qquad (4.5)$$

[11] See Appendix 3.

[12] In Appendix 4 it will be shown that this equation is equivalent to the Papapetrou equation.

This equation conserves the norm of $\mathbf{S}$. Indeed, $\dfrac{D\mathbf{S}^2}{d\tau} = 2\mathbf{S}\cdot\dfrac{D\mathbf{S}}{d\tau}$. However, when $\mathbf{S}$ satisfies (4.5) it is only the longitudinal component of the absolute derivative of $\mathbf{S}$ which can differ from zero. But, according to $\mathbf{S}\cdot\mathbf{V} = 0$ this component being parallel to $\mathbf{V}$ has zero scalar product with $\mathbf{S}$. Hence $\mathbf{S}^2 = -\vec{S}^2 = constant$.

When $\mathcal{L}$ is a geodetic the derivative of the condition $\mathbf{S}\cdot\mathbf{V} = 0$ leads to $\mathbf{V}\cdot\dfrac{D\mathbf{U}}{d\tau} = 0$. This means that the longitudinal component of the absolute derivative of $\mathbf{S}$ vanishes too. Therefore, in this case (4.5) is equivalent to the equation of parallel transportation.

Having the equation (4.5) at hand we can now calculate Ω for the precession of gyroscopes revolving on a circular trajectory $r = const$ with an arbitrary angular velocity ω:

$$\Omega = \omega\frac{1-\frac{3}{2}\frac{r_g}{r}}{\sqrt{1-\frac{r_g}{r}-\left(\frac{r\omega}{c}\right)^2}}. \tag{4.6}$$

This formula[13] gives Ω measured in coordinate time and referred to the local coordinate base. The quantities r and ω can be chosen independently but when the gyroscope revolves on a circular *geodetic* orbit they are related by (3.1) to each other. In this case (4.6) reduces to (3.2) as should be. The precession rate is given again by

$$w = \omega - \Omega = \omega\left(1-\frac{1-\frac{3}{2}\frac{r_g}{r}}{\sqrt{1-\frac{r_g}{r}-\left(\frac{r\omega}{c}\right)^2}}\right). \tag{4.7}$$

When r is the radius of the Earth and $\omega = 2\pi\ rad/day$ then (4.7) is equal to the precession rate of a gyroscope resting on the Equator[14]. The precession of the gyroscope's axis takes place in the equatorial plane perpendicular to the ground. But when we observe the precession of a gyroscope in our laboratory on the ground we perceive directly Ω rather then w (in the absence of geodetic precession we would experience rotation of angular speed $360°/day$ opposite to the rotation of the Earth). Furthemore our laboratory clocks show proper time τ which is related to coordinate time as

$$d\tau = \frac{1}{c}\sqrt{(1-r_g/r)c^2dt^2 - r^2d\varphi^2} = dt\cdot\sqrt{1-\ r_g/r - r^2\omega^2/c^2}. \tag{4.8}$$

Therefore the proper time angular velocity Ω_* observed in the laboratory is given by the equation

$$\Omega_* = \Omega\frac{dt}{d\tau} = \omega\frac{1-\frac{3}{2}\frac{r_g}{r}}{1-\frac{r_g}{r}-\left(\frac{r\omega}{c}\right)^2}. \tag{4.9}$$

[13]The proof of (4.6) is outlined in Appendix 5.

[14]For the sake of simplicity precession of the axis of the Earth's rotation is neglected.

Equation (4.7) can be applied also to a test gyroscope mounted on a turntable at a distance r from the axis of rotation. In this case gravitation plays no role and we should put $r_g = 0$. Then

$$w_T = \omega \left(1 - 1/\sqrt{1 - v^2/c^2} \right), \qquad (v = r\omega) \tag{4.10}$$

which is the well-know formula for the *Thomas-precession.*

5 Geodetic precession as seen by the cosmonauts.

Ω is equal to the angular speed of the spacecraft's rotation with respect to the local coordinate base the element $\mathbf{e}_{(r)}$ of which points opposite the direction of the central body. Therefore, Ω can be thought of as the angular velocity with respect to this body and the cosmonauts would observe the central body to revolve around them with this speed were their clocks showing the coordinate time t. But the time shown by these clocks is the proper time τ (the coordinate time in the Fermi coordinate system) and the observed angular momentum is Ω_* given by (4.9) rather than Ω. If we substitute here the angular velocity ω as given by (3.1) we arrive at the surprising equality

$$\Omega_* = \omega. \tag{5.1}$$

The conclusion is, therefore, that as a consequence of the geodetic precession which is of positive (prograde) direction the angular speed of the revolution of the central body around the spacecraft as measured in t is smaller than ω but this difference is *precisely compensated* by the proper time correction. It can be shown that a completely analogous compensation takes place in Kerr-metric too for circular geodetics in the equatorial plane. It would be interesting to know whether the equality (5.1) has some sensible geometric meaning or is a mere coincidence.

The equality (5.1) is independent of the choice of the coordinate system because it can be experimentally verified. For the sake of definiteness consider now the revolution around the Sun (instead of the Earth) and neglect the effect of the planets and Sun's rotation on the spacecraft. Ω_* is an invariant since it is measured by the proper time interval in which the Sun makes a complete revolution (2π radian) around the spacecraft. The angular speed ω is measured in Schwarzschild coordinate time but this coordinate time is identical to the proper time of the distant observer. Imagine a perpendicular to the spacecraft's plane of revolution at its centre and a monitor spacecraft which is at rest in a distant point on this perpendicular (this can be achieved by a weak thrust in the direction opposite to the Sun). From this monitor it is in principle possible to observe the time Δt in which the spacecraft makes a complete circle. From this time interval ω can be calculated. Now a telephon call from the monitor to the spacecraft is sufficient to decide whether ω is equal to Ω_* or not.

As seen from (3.2) Ω decreases when r becomes smaller and smaller and becomes zero at $r = 3r_g/2$. At this radius $w = \omega$ and the spacecraft turns the same side toward the central body throughout (as the Moon does with the Earth). But this situation cannot actually be reached because the geodetic $r = 3r_g/2$ is lightlike $\big($see (4.8) and (3.2)$\big)$ and no spacecraft can revolve along it. One would expect that approaching this limiting radius the cosmonauts would find the central body to revolve around them slower and slower but just the opposite happens. With decreasing r the angular speed ω increases and as a consequence of (5.1) Ω_* increases with it too. Its limiting value is $2/\sqrt{27} \times c/r_g$.

6 Geodetic precession around a rotating celestial body.

The formula for the geodetic precession around a rotating central body contains a contribution proportional to the angular momentum J of the body which is called *drag* (or *frame dragging*). The term 'geodetic precession' is retained for the contribution independent of J but this practice may lead to ambiguity because the phenomenon as a whole is a geodetic precession too.

In the GP-B experiment the central body was the Earth and the drag was by about two orders of magnitude slower then geodetic precession proper. Hence its only component to be taken into consideration is that whose axis is perpendicular to the axis of geodetic precession.

To have a glimps of the direction of the drag's axis we may 'switch off' the orbital motion and consider the precession of gyroscopes at rest in Kerr spacetime which is 'pure drag'. The trajectory in this case is not a geodetic and so the formula (4.5) should be used in the calculation. The result is summarized on Figure 2 on which the direction and the length of the arrows illustrate the orientation of the drag's axis and its angular speed

$$\Omega_{drag} = \frac{ar_g c}{2r^3}\sqrt{1 + 3\cos^2\vartheta}$$

$(J = Mca)$. The picture behind the term 'drag' is that the rotation of the central body drags with itself the local inertial frames which are rotation-free in the sense of Chapter 1. As we see this metaphor is pertinent only along the axis of rotation because in the equatorial plane rotation-free spacecrafts precess opposite to the central body[15].

[15] As it was pointed out by L. Schiff 'drag' would be truely appropriate term only within the framework of the ether hypotheses. If ether was a Newtonian fluid whose relative velocity at the surface of the rotating central body is zero and it would be the ether which would drag the rotation-free spacecraft with itself then the arrows on Figure 2 would conform with intuition.

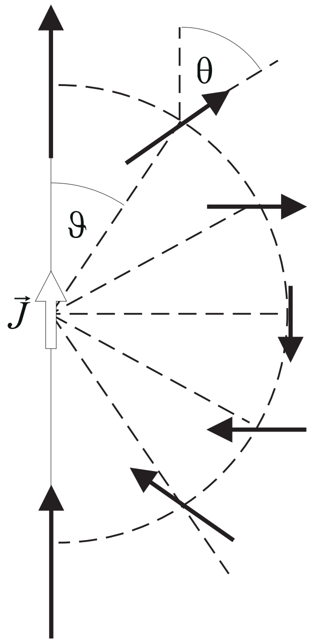

Figure 2

The trajectory of the GP-B capsule lay in a polar plane. Therefore, Fig. 2 may be assumed to show the drag axes at different points of the trajectory. The drag axes being in the trajectory's plane the drag *moves out* gyroscopes' axes from that plane. The geodetic precession proper, on the contrary, leaves these axes within this plane. The orthogonality of these two contributions is essential for the separate measurement of geodetic precession proper and drag[16].

7 Geodetic precession with respect to sky.

As we have discussed in Chapter 2 precession of the rotation-free satellites can in principle be observed from the Earth's surface. But accuracy of this kind of observations is far too low for measuring the extremly slow angular velocity of geodetic precession. The necessary precision can be ensured by astronomical methods only referring the orientation of the gyroscopes carried by the satellite to a *guide star*. But this is an acceptable method only if we are sure that the observed direction of the guide star remains during measurement fixed with respect to those directions which would be employed in a procedure confined to Solar System.

[16]"We measured a geodetic precession of 6.600 plus or minus 0.017 arcseconds and a frame dragging effect of 0.039 plus or minus 0.007 arcseconds," said Stanford University physicist Francis Everitt, principal investigator of the Gravity Probe B mission, on the May 4, 2011 press conference of NASA. These data are in excellent agreement with theory (see Chapt.8)

The proper motion of the star can be simply eliminated: A distant quasar must be chosen for a guide star. But a problem of principle remains. The criterion of the absence of rotation formulated in Chapt.1 contains no reference at all to starry sky (to the whole of Cosmos). Is it not possible that skies do in fact rotate with respect to the local reference frames which are in this sense non-rotating? If they indeed do then the angular velocity with respect to the guide star need not coincide with the speed of the geodetic precession (and that of the drag) calculated within, say, the Solar system (or in the environment of the Earth).

For convenience we can distinguish the following two types of absence of rotation:

1. A spacecraft (local inertial frame) is rotation-free *in the sence of Foucault* if no Coriolis-force is acting in it. This is the criterion formulated in Chapter 1.

2. A spacecraft (local inertial frame) is rotation-free *in Copernican sense* if its orientation is not changing with respect to the sky.

These two criteria of rotationlessness are logically independent of each other and so a local inertial frame may rotate with different angular velocities in the two senses; it may be even rotation-free in one sense and rotating in the other. It is, therefore, by no means inevitable that the plane of the Foucault-pendulum which is acted upon by the Coriolis force must at the Poles rotate with exactly the same angular velocity as the sky does. Hence, the empirical equality (with some accuracy) of these angular velocities is not a new independent proof of the Earth's rotation with respect to skies with a period of 24 hours but rather it is an indication that *empirically* the two concepts of rotation are apparently equivalent to each other.

Since in the Newtonian physics inertial frames are of infinite spatial extension the equality of the two angular velocities leads to the conclusion that the reference frame which is not rotating with respect to the sky is an inertial frame i.e. *inertial frames are determined* (at least from the point of view of their rotation) *by the fixed stars.* And, remembering Chapter 2, the corollary is that spacecrafts which do not rotate in the sense of Foucault are automatically rotation-free in the Copernican sense too.

But according to general relativity this conclusion is erroneous. Spaccrafts orbiting freely without rotation (in the sense of Foucault) on different trajectories perform geodetic precession with various angular velocities and it is impossible that they all have fixed orientation with respect to skies (i.e. neither of them rotates in Copernican sense).

Therefore, the answer to our question — 'Is it not possible that skies do rotate with respect to the local reference frames which are in the sense of Chapter 1 non-rotating?' — is that this is possible and according to general relativity just this happens. This aspect of the geodetic precession concerns the very foundations of our physical world view.

But these considerations are still compatible with the possibility that the angular velocity of the geodetic precession with respect to fixed stars is equal to the same w which was cited in Chapt.3 and is based on the notion of rotationlessness of Chapt.1. The condition for this coincidence is that in coordinate system of Chapt.3 (Schwarzschild coordinates around the central body) the light rays from distant stars be of constant direction. One is inclined to take this requirement for a triviality but it is not (see Chapt.10) though this conviction is strongly supported by the *empirical fact* that planetary orbits calculated 'locally' (i.e. in Schwarzschild coordinates) are apparently correct also with respect to the spherical coordinate system defined by the fixed stars. It is this agreement which permits the determination of the angular velocity of geodetic precession with respect to a suitably chosen guide star.

8 The GP-B experiment.

In the Gravity Probe B experiment[17] of NASA the spacecraft carrying four independent gyroscopes orbited the Earth on a polar circle at 642 km altitude. The spacecraft was lunched at 20 april 2004 after decades of careful preparations. Data collection lasted 352 days (with occasional interruptions).

The gyroscopes' spins were initially oriented along the symmetry axis of the spacecraft (an elongated cigar) which coincided with the optical axis of a telescope mounted on the end of the spacecraft. The latter was 'steered' by means of attitude control forces to keep the centroid of the image of the guide star which was seen in the orbital plane on the telescope axis.

The precession of the gyroscopes' axes took place with respect to this direction. The angle of deflection due to the Sun and the rotating Earth was calculated in first post-Newtonian approximation with the following result:

— 6600 milliarcsecs in North-South direction (geodetic precession proper).

— 42 milliarcsecs in East-West direction (drag).

The calculation consists of the following main steps:

1. Determination of the trajectory (the geodetic $\mathcal{G}$) from the post-Newtonian metric and observations.

2. The light rays from the guide star cover the domain of spacetime occupied by $\mathcal{G}$. The tangent vectors of this congruence make a vector field $\mathbf{k}$ ($\mathbf{k}^2 = 0$) which is homogeneous only asymprorically toward the guide star. The departure from homogeneity (light deflection) makes it necessary to calculate this vector field in the post-Newtonian metric.

3. The direction of the gyroscopes' axes define a vector field $\mathbf{U}$ along $\mathcal{G}$ ($\mathbf{U}^2 = -1$, $\mathbf{U} \cdot \mathbf{V} = 0$) which can be calculated using (1.3).

[17] C. M. Will, Phys. Rev. **D67**, 062003 (2003).

4. The main target of the experiment was to determine the angle of deflection of the gyroscopes' axes from the symmetry axis of the spacecraft. The attitude control system kept the symmetry axis in the direction of the guide star

$$\hat{\mathbf{k}} = \frac{c}{\mathbf{V}\cdot\mathbf{k}}\,\mathbf{k} - \frac{1}{c}\mathbf{V} \qquad (\hat{\mathbf{k}}\cdot\mathbf{V} = 0, \quad \hat{\mathbf{k}}^2 = -1)$$

which is the projection of $\mathbf{k}$ along $\mathbf{V}$ on the hypersurface spanned by the spacelike elements $(\mathbf{e}_{(X)},\ \mathbf{e}_{(Y)},\ \mathbf{e}_{(Z)})$ of the Fermi-basis. The telescope is oriented of course in the direction $-\hat{\mathbf{k}}$. From the observations the time dependence of the scalar product $\hat{\mathbf{k}}\cdot\mathbf{U}$ can be deduced.

The quantity $\hat{\mathbf{k}}\cdot\mathbf{U}$ contains the contribution of the aberration too which is caused by the motion of the spacecraft on $\mathcal{G}$. It is an effect much larger than geodetic precession and drag but since it can be calculated with a high accuracy it can be utilized for purposes of calibration.

9 The Mach-problem in Newtonian physics.

When one thinks of the observational verification of the Newtonian theory of planetary motion one tacitly assumes that the predictions of the theory are true both with respect to the inertial frame in which the calculations have been performed and in the spherical coordinate system which is defined by the fixed stars. Experience supports this assumption. I will call *Mach-problem* the question of why is it so.

In Newtonian physics Mach-problem is equivalent to the question of why inertial frames selected by *local* experiences are not rotating with respect to skies. Or, using the terminology of Chapt.7, why Copernican rotation is indistinguishable from the rotation in the sense of Foucault. The question was asked by Newton himself in connection with his famous ecperiment with the rotating bucket: How can it be that the surface of the water rotating together with the bucket becomes flat in just that state when the bucket is at rest with respect to the fixed stars?

Newton found the explanation in the notion of the *absolute space*. Let us denote a certain point of space at a given moment of time by P. The space is absolute if we can point at least in priciple *to the same* point P at any later moment of time. In other words the points of the absolute space are capable to retain their self-identity. Points belonging to material media do possess this property and so absolute space from this point of view is 'medium like' though from other points of view it is not: It for example does not exert drag on moving bodies.

If space is absolute then rest and motion are absolute too: A point mass is at rest if it remains in the same point of space. This principle can be extended to rotation: A body is not rotating if its orientation is not changing with respect to the absolute space. If we add to this the rather natural assumption that fixed stars are at rest in the absolute space then we no longer need to wonder that bodies not rotating according to local criteria are at rest with respect to skies too.

But the notion of the absolute geometric space with its medium-like properties is foreign to modern physics. Another objection against it is that it is applicable only to rotation and does not select among uniformly moving reference frames the only one which is at rest.

It was Ernst Mach who emphasized that the absolute or relative conception of space is not a question of taste because it depends on the underlying physical theory: In Newtonian physics rotation is inevitably absolute. Indeed, if in the otherwise empty world we imagine a reference frame in which physical experiment can be performed than by the presence or absence of the inertial forces we can decide whether our reference frame is rotating or not. Let us cite Mach himself:

> We are allowed to think freely of Earth as rotating around its axis or as beeing at rest and the fixed starts rotate around it. *Geometrically* these two cases are indistinguishable from each other. But if we decide that the Earth is at rest and the fixed stars rotate around it then where the Equatorial bulge, the Foucault-experiment and many other similar phenomena come from if we accept the law of inertia is valid in its present form? This difficulty can be overcome by two ways. Either all kinds of motion are absolute or we do not understand properly the law of inertia. I prefer the second possibility. To make use of it we must find that formulation of the law of inertia which leads to the same results from both points of view considered above. This can be done only if to take into account masses all around the Universe.

According to this program called *Mach-principle* inertial forces are actually true forces exerted upon moving bodies by the distant masses of the Universe. No sufficiently elaborated theory of this kind has so far been proposed.

Immediately after the birth of general relativity the question arose as to the validity of the Mach-principle in the new theory. Already in 1918 H. Thirring and J. Lense proved in weak field approximation that a local inertial frame at the center of a rotating spherical shell does rotate with the angular velocity $\Omega = 4GM/3c^2R \times \omega$ in which ω, R and M are the angular velocity, radius and mass of the shell respectively. If this model were sufficiently close to reality its parameters could be chosen so as to make the equality $\Omega = \omega$ valid. This equality would state that among the local reference frames the inertial one is that which does not rotate with respect to the shell. This would provide proper basis for the empirical fact that inertial frames do not rotate with respect to the fixed stars. But the model is far too unrealistic from a cosmological point of view and today it has historical significance only as the first recognition of frame dragging[18].

[18] In 1917 Einstein in his fundamental work on cosmology supposed that in a theory of this

10 The Mach-problem in general relativity.

General relativity theory inherited the Mach-problem from its predecessor because Solar System is treated in it in isolation from the rest of the Cosmos while planetal orbits obtained from such calculations turn out valid also with respect to the spherical coordinates fixed by distant heavenly bodies. This empirical fact which is fully exploited in GP-B experiment needs explanation even though no doubt arises as for its validity to a high degree of accuracy.

Mach-problem concerns the connection between the spacetime around a solitary star — the Sun — on the one hand and the cosmological solution on the other. In 1946 Einstein and E. Straus put the problem into the following form[19]. Choose the center of the Sun for the origin of the cosmological coordinate system and imagine a two dimensional sphere around it whose points have fixed cosmological coordinates (i.e. the size of the sphere expands together with the Universe). In spacetime this two-sphere is a boundary of a three dimensional tube Σ whose axis is the world line of the center of the Sun. Is it possible to match the interior Schwarzschild solution defined by the mass of the Sun with the cosmological solution outside?

Einstein and Straus used weak field approximation in the vicinity of Σ to compare the two solutions and found smooth matching possible. They concluded from this that the expansion of the Universe should not have influence on local phenomena within the domain of the Solar System. Later E. Schücking reconsidered the problem and proved the possibility of matching metrics without any approximation[20]. That was not a simple task because of the necessity to transcribe the Robertson-Walker metric to *curvature coordinates* in which the angular part of the cosmological solution is of the form $r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$. Moreover the Schwarzschild coordinate time had to be suitably rescaled. Schüking concluded that matching is possible only if the mass omitted from the cosmological solution by excluding the interior of Σ from it is exactly equal to the mass of the Sun.

The positive solution of the Einstein-Straus problem is an indication that Mach-problem can probably be settled in this way provided the proof can be extended to the Kerr-metric too which is by no means certain[21]

The Einstein-Strauss approach may itself raise doubts. Since mass in the

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kind Mach-principle is expressed in the dependence of the gravitational constant G on the mass and characteristic size of the Universe. On dimensional ground such a relation should be of the form $G = const \times c^2 R/M$. Einstein's view of the Universe as a sphere of radius R was motivated by this expectation. But field equations possessed this type of solution only if a new term containing the cosmological constant Λ was introduced into them which determined R as $R = 1/\sqrt{\Lambda}$. So either R or Λ could be considered a characteristic length but according to the present state of art it is the latter which is likely to perform this role.

[19] A. Einstein and E. Straus, Rev. Mod. Phys. **17**, 120 (1945) and **18**, 148 (1946)

[20] E. Schüking Z. Phys. **137**, 595 (1954). The continuity of the normal derivative along Σ remained outside the scope of the paper. The result of this paper was later generalized to nonzero cosmological constant in R. Balbinot el al, Phys. Rev. **D38**, 2415 (1988)).

[21] Reading Schüking's paper it is a rather striking experience to see as in the course of coordinate transformations the characteristic ingredient $(1-2m/r)$ of the Schwarzschild metric gradually emerges in the cosmological solution.

Universe is contained in isolated heavenly bodies rather than being smoothly distributed matching of the Schwarzschild solution to the smoothed Robertson-Walker metric may have no physical meaning at all.

Appenddix 1: The torque acting on the gyroscope.

The torque $\mathbf{K}$ which in a reference frame rotating with the angular velocity $\boldsymbol{\omega}$ acts on an isolated gyroscope is equal to the sum of the torques the Coriolis-force $-2m(\boldsymbol{\omega}\times\mathbf{v})$ exerts on the parts of the gyroscope:

$$\mathbf{K} = -2\sum m_i\big(\mathbf{r}_i \times (\boldsymbol{\omega}\times\mathbf{v}_i)\big) = 2\sum m_i(\mathbf{r}_i\cdot\boldsymbol{\omega})\mathbf{v}_i. \tag{A}$$

The index i of summation refers to the parts of the gyroscope. If the angular velocity of the gyroscope is $\boldsymbol{\Omega}$ ($\boldsymbol{\Omega}\gg\boldsymbol{\omega}$) then

$$\mathbf{v}_i = (\boldsymbol{\Omega}\times\mathbf{r}_i)\perp\mathbf{r}_i. \tag{B}$$

Substituting this into (A) we obtain

$$\mathbf{K} = 2(\boldsymbol{\Omega}\times\mathbf{V}) \tag{C}$$

where

$$\mathbf{V} = \sum m(\mathbf{r}_i\cdot\boldsymbol{\omega})\mathbf{r}_i.$$

In the coordinate system $OXYZ$ of the principal axes of inertia we have

$$V_\alpha = \sum_\beta\left(\sum m_i r_{i\alpha} r_{i\beta}\right)\omega_\beta \qquad (\alpha,\ \beta = x,\ y,\ z).$$

According the definition

$$I_{\alpha\beta} = \sum m r_i^2\cdot\delta_{\alpha\beta} - \sum m_i r_{i\alpha} r_{i\beta} \tag{D}$$

of the tensor of inertia the r.h.s. of the previous equation can ve written as

$$V_\alpha = \sum_\beta\left(\sum m r_i^2\cdot\delta_{\alpha\beta} - I_{\alpha\beta}\right)\omega_\beta. \tag{E}$$

The inertia tensor is diagonal with the principal moments of inertia $I_x,\ I_y,\ I_z$ and so we obtain for the diagonal sum (trace) of (D) the expression

$$\sum m_i r_i^2 = \frac{1}{2}(I_x + I_y + I_z)$$

which can be substituted into (E):

$$V_x = \frac{1}{2}(I_y + I_z - I_x)\omega_x, \qquad V_y = \frac{1}{2}(I_z + I_x - I_y)\omega_y, \qquad V_z = \frac{1}{2}(I_x + I_y - I_z)\omega_z.$$

For a spherically symmetric gyroscope $I_x = I_y = I_z = I$, hence $\mathbf{V} = \frac{1}{2} I\boldsymbol{\omega}$. Using this in (C) and taking into account the formula $\mathbf{J} = I\boldsymbol{\Omega}$ for the angular momentum of the gyroscope, we obtain for the equation of motion the formula

$$\dot{\mathbf{J}} = \mathbf{K} = (-\boldsymbol{\omega}) \times \mathbf{J}.$$

For our purposes the interpretation of this relation is that if a spherically symmetric gyroscope is isolated (no true torque acts on it) and its axis of rotation precesses with the angular velocity $-\boldsymbol{\omega}$ then the reference frame in which this precession is observed rotates with the angular velocity $\boldsymbol{\omega}$ (i.e. it is not an inertial frame).

Appendix 2: The Fermi coordinate system.

To each timelike geodetic $\mathcal{G}$ a special coordinate system $\mathcal{K}_{\mathcal{G}}$ named after *Enrico Fermi* can be attached. From the physical point of view its distinguishing feature is that in the vicinity of $\mathcal{K}_{\mathcal{G}}$ (i.e within the spacecraft) it is practically Minkowskian in full agreement with the equivalence principle. As a consequence, if astronauts in spacecrafts on different orbits measure light velocity using, say, Fizeau's method of rotating mirrors, obtain the same value for it in any direction with respect to the spacecraft. As it has already been explained in the main text translation of this property to mathematics is expressed by the following conditions:

1. $g_{ij}(\mathcal{G}) = \eta_{ij}, \qquad \Gamma^i_{jk}(\mathcal{G}) = 0;$
2. The coordinate time of $\mathcal{K}_{\mathcal{G}}$ must coincide (up to an irrelevant constant) with the proper time τ of $\mathcal{G}$: $dt = d\tau$.

It is the vanishing of Γ^i_{jk} (or $\frac{\partial g_{ij}}{\partial x^k}$) on $\mathcal{G}$ which makes the proof of the statement a nontrivial task.

The construction of $\mathcal{K}_{\mathcal{G}}$ begins with the choice of its basis $\mathbf{e}_{(A)}$ $(A = 0, 1, 2, 3)$ on $\mathcal{G}$.

In order to satisfy the second requirement the element $\mathbf{e}_{(0)}$ must coincide with the normalized tangent vector to $\mathcal{G}$ (i.e. the four-velocity $\mathbf{V}$ of the body moving on $\mathcal{G}$ divided by c):

$$\mathbf{e}_{(0)} = \frac{1}{c}\mathbf{V}$$

(the basis is *line oriented*). Since in this case

$$1 = e^0_{(0)}(\tau) = \frac{1}{c}V^0(\tau) = \frac{dt}{d\tau}, \tag{A}$$

equality of t and τ on $\mathcal{G}$ is indeed ensured.

Select now an arbitrary reference point Q on $\mathcal{G}$ where $\tau = 0$ and complete $\mathbf{e}_{(0)}(0) = \mathbf{V}(0)/c$ to a basis $\mathbf{e}_{(A)}(0)$ by adding three arbitrarily chosen orthonormal spacelike unit vector $\mathbf{e}_{(\Lambda)}(0)$ to it ($\Lambda = 1, 2, 3$). Then $\mathbf{e}_{(A)}(0)$ can be extended to a basis $\mathbf{e}_{(A)}(\tau)$ along the whole $\mathcal{G}$ by parallel transportation. Requirement (A) is then obviously satisfied. Moreover, due to the relation $g_{AB}(\tau) = (\mathbf{e}_A(\tau) \cdot \mathbf{e}_B(\tau)) = \eta_{AB}$ the first of condition 1. is also fulfilled.

In order to exhibit the vanishing of the connection coefficients on $\mathcal{G}$ the construction of $\mathcal{K}_{\mathcal{G}}$ must be accomplished by the following prescription of coordinates to an arbitrary point P of space-time.

Draw a spacelike geodetic $\mathcal{H}$ through P which intersects $\mathcal{G}$ orthogonally at the point $\mathcal{G}(\tau)$ which will be denoted also by O. Parametrize the geodetic $\mathcal{H}$ by the distance of its points from O and denote the distance of P from O by r.

Let $\mathbf{v}$ be the tangent vector of $\mathcal{H}$ at O. Since $\mathbf{v} \cdot \mathbf{e}_0(\tau) = 0$ by definition the vector $\mathbf{v}$ is spacelike and it will be normalized to unity: $\mathbf{v}^2 = -1$. If its direction cosines are denoted by c_Λ then

$$\mathbf{v} = c_1 \mathbf{e}_{(1)}(\tau) + c_2 \mathbf{e}_{(2)}(\tau) + c_3 \mathbf{e}_{(3)}(\tau).$$

$\mathcal{K}_{\mathcal{G}}$ can now be defined by assigning to P the coordinates

$$x^A = (x^0,\, x^1,\, x^2,\, x^3) = (c\tau,\, c_1 r,\, c_2 r,\, c_3 r),$$

where, as usual, $x^0 = ct$. The coordinate system $\mathcal{K}_{\mathcal{G}}$ covers that part of the space-time where this procedure is unique.

In order to prove that in the coordinate system just defined $\Gamma^i_{jk}(\mathcal{G}) = 0$ we notice that $\mathcal{H}$, according to its definition, satisfies the geodetic equation

$$\frac{d^2 x^A}{dr^2} + \Gamma^A_{BC} \frac{dx^B}{dr} \frac{dx^C}{dr} = 0,$$

which for $x^0 = ct$ and $x^\Lambda = c_\Lambda r$ reduces to

$$\Gamma^A_{\Delta\Lambda} c_\Delta c_\Lambda = 0.$$

Since on $\mathcal{G}$ this equation must be satisfied for any possible triple of direction cosines smoothness of the manifold ensures that $\longrightarrow 0$ as $r \longrightarrow 0$ i.e. $P \to \mathcal{G}$.

It still remains to consider connection coeffients of the type Γ^A_{0B}. To this end we notice that so far no use has been made of the fact that $\mathcal{G}$ is a geodetic and the basis $\mathbf{e}_{(A)}(\tau)$ on it has been constructed by parallel transportation of $\mathbf{e}_{(A)}(0)$. As a consequence the *absolute derivative* $D/d\tau$ along $\mathcal{G}$ of the basis elements vanish:

$$\frac{De^B_{(A)}(\tau)}{d\tau} \equiv \frac{de^B_{(A)}(\tau)}{d\tau} + \Gamma^B_{CD}(\tau) V^C(\tau) e^D_{(A)}(\tau) = 0. \tag{B}$$

But on $\mathcal{G}$ we have $e^B_{(A)}(\tau) = \delta^B_A$ and $V^C(\tau) = c e^C_{(0)}(\tau) = c\delta^C_0$ and the above equation of parallel transportation reduces to the statement that connection coefficients with at least one lower index of zero value must all vanish.

The existence of Fermi coordinates on pseudo-Riemannian manifolds expresses the *compatibility* of the pseudoriemannian geometry with the strong equivalence principle: The natural coordinate system with respect to which a nonrotating freely orbiting spacecraft of limited spatial extension is at rest coincides with the Fermi coordinates. In such spacecrafts the axis of rotation of any gyroscope aimes permanently at one and the same point on the capsule's wall. The joint conclusion from these observations is that the four spin vector **S** of a freely orbiting gyroscope is parallel transported along $\mathcal{G}$, i.e. its absolute derivative is equal to zero:

$$\frac{DS^A(\tau)}{d\tau} \equiv \frac{dS^A(\tau)}{d\tau} + \Gamma^A_{BC}(\tau)V^B(\tau)S^C(\tau) = 0. \tag{C}$$

Now in physics rotationless motion is conceptually identified with the motion of the axes of freely moving spherical gyroscopes. As we have just seen *on geodetics* this criterion is satisfied by parallel transportation.

It is expedient to identify those connection coefficients which are responsible for the Coriolis-force since it is the vanishing of just them which makes rotationless motion of isolated gyroscopes possible (see Appendix 1).

Consider a spacecraft which rotates with constant angular velocity ω while moving along $\mathcal{G}$. If its axis of rotation is the z-axis of $\mathcal{K}_\mathcal{G}$ then it will be at rest with respect to the primed coordinate system $\mathcal{K}'_\mathcal{G}$ defined by the equations

$$t' = t, \quad x' = x\cos\omega t + y\sin\omega t, \quad y' = -x\sin\omega t + y\cos\omega t, \quad z' = z. \tag{D}$$

On $\mathcal{G}$ the general formula

$$\Gamma^{i'}_{j'k'} = \frac{\partial x^{i'}}{\partial x^l}\frac{\partial x^m}{\partial x^{j'}}\frac{\partial x^n}{\partial x^{k'}}\Gamma^l_{\ mn} + \frac{\partial x^{i'}}{\partial x^m}\frac{\partial^2 x^m}{\partial x^{j'}\partial x^{k'}} \tag{E}$$

of the transzformation of the connection reduces to its second term and the only nonzero primed connection coefficients on $\mathcal{G}$ are

$$\Gamma^{y'}_{x't'}(\mathcal{G}) = -\Gamma^{x'}_{y't'}(\mathcal{G}) = \omega. \tag{F}$$

Therefore, the geodetic equation for a slowly moving point mass in $\mathcal{K}'_\mathcal{G}$ is

$$\frac{d^2x'}{dt^2} = -2\Gamma^{x'}_{y't'}\frac{dy'}{dt} = 2\omega v_{y'}, \qquad \frac{d^2y'}{dt^2} = -2\Gamma^{y'}_{x't'}\frac{dx'}{dt} = -2\omega v_{x'}.$$

Multiplied by the mass these are the Newtonian equations of motion under the influence of the Coriolis-force. Therefore, connection coefficients of the type $\Gamma^\Delta_{0\Lambda}$ $(\Delta \neq \Lambda)$are the trademarks of the Coriolis-force whose presence makes spherical gyroscopes' axes to rotate.

Appendix 3: The Fermi-Walker derivative.

In the previous Appendix we discussed the rules of attaching Fermi coordinate system $\mathcal{K}_\mathcal{G}$ to *freely orbiting* spacecrafts or gyroscopes with respect to which their position and orientation remains unchanged (see Fig. 1). But when these objects are *accelerating* they do not move on geodetics any more but their trajectory becomes a general timelike world line $\mathcal{L}$. How to attach Fermi coordinates to moving objects in this more general case?

Fermi coordinates belonging to $\mathcal{L}$ will be naturally denoted by $\mathcal{K}_{\mathcal{L}}$. The necessary modification in constructing them concerns the method of spreading over $\mathcal{L}$ of the line oriented pseudo orthonormal tetrad $\mathbf{e}_{(A)}(0)$ chosen at the reference point Q. As explained earlier the equality of the Fermi coordinate time t to the proper time τ of the object moving on $\mathcal{L}$ is ensured only if $\mathbf{e}_{(0)}(\tau)$ remains everywhere equal to $\mathbf{V}(\tau)/c$. *On a geodetic* this property is obviously warranted provided the absolute derivative $D/d\tau$ of $\mathbf{e}_{(A)}(\tau)$ vanishes along the word line since $\mathbf{V}(\tau)$ obeys this same condition. Therefore,

$$\frac{D\mathbf{e}_{(A)}(\tau)}{d\tau} = 0. \tag{A}$$

The tangent vector $\mathbf{V}$ of a general $\mathcal{L}$ does not satisfy the geodetic equation $D\mathbf{V}/d\tau = 0$. Hence, the tetrad basis calculated from (A) will break the equality $\mathbf{e}_{(0)}(\tau) = \mathbf{V}(\tau)/c$ at $\tau \neq 0$. The objective of this Appendix is, therefore, to find a generalization ${}^*D/d\tau$ of the absolute derivative which by means of the modidfied equation

$$\frac{{}^*D\mathbf{e}_{(A)}(\tau)}{d\tau} = 0. \tag{B}$$

would provide a correct line oriented basis $\mathbf{e}_{(A)}(\tau)$ throughout $\mathcal{L}$.

Further specification of the operation ${}^*D/d\tau$ concerns the fact that bodies can move on a given trajectory rotating or without rotation. This is so in both newtonian and relativistic mechanics, but in relativity the distinction between these two cases is less obvious and, therefore, more interesting then in elementary newtonian physics. It is, therefore, expedient to impose on $\mathcal{K}_{\mathcal{L}}$ the condition that objects with fixed orientation in it should move *without rotation.* In geodetic motion parallel transportation (A) of the basis satisfies this condition authomatically.

In order to find the covariant expression of the operation ${}^*D/d\tau$ we notice that *in the coordinate system* $\mathcal{K}_{\mathcal{L}}$ it obviouly reduces to simple differentiation:

$$\frac{{}^*D\mathbf{U}}{d\tau} \stackrel{*}{=} \frac{d\mathbf{U}}{dt}. \tag{C}$$

The star on the sign of equality is the reminder that the equation is valid exclusively in the coordinate system $\mathcal{K}_{\mathcal{L}}$.

In (C) equality of τ and t is expressed explicitely. If, moreover, the right hand side is set equal to zero then the resulting equation expresses non-rotational propagation of the four vector $\mathbf{U}$ along $\mathcal{L}$ because in $\mathcal{K}_{\mathcal{L}}$ the spatial triple $\mathbf{e}_{(\Lambda)}(\tau)$ of the basis $\mathbf{e}_{(A)}(\tau)$ is by definition non-rotating. The definition (C) also ensures that the derivation ${}^*D/d\tau$ obeys Leibniz-rule.

Transcription of the r. h. s. of (C) to covariant form starts with the expression of it through the absolute derivative of **U**:

$$\frac{dU^A}{dt} \stackrel{*}{=} \frac{dU^A}{d\tau} = \frac{DU^A}{d\tau} - \Gamma^A_{BC} V^B U^C.$$

Hence

$$\frac{{}^*DU^A}{d\tau} \stackrel{*}{=} \frac{DU^A}{d\tau} - \Gamma^A_{BC} V^B U^C. \tag{D}$$

The first term is already covariant but the connection coefficients in the second term still belong to $\mathcal{K}_{\mathcal{L}}$. But when $\mathcal{L}$ is a geodetic the second term is absent. These Γ-s must, therefore, be either proportional to the acceleration or to vanish.

Since construction of $\mathcal{K}_{\mathcal{L}}$ is identical to that of $\mathcal{K}_{\mathcal{G}}$ vanishing of the connection coefficients of the type $\Gamma^A_{\Delta\Lambda}(\mathcal{L})$ can be proven in the same way as in Appendix 2. Moreover Christoffel formula for the connection coefficients leads in $\mathcal{K}_{\mathcal{L}}$ to the equalities

$$\Gamma^O_{OO}(\mathcal{L}) \stackrel{*}{=} \Gamma^X_{OX}(\mathcal{L}) \stackrel{*}{=} \Gamma^Y_{OY}(\mathcal{L}) \stackrel{*}{=} \Gamma^Z_{OZ}(\mathcal{L}) \stackrel{*}{=} 0.$$

The only remaining connection coefficients are those which contain exactly two zeros. But they are pairwise equal to each other. Indeed, if in the equation

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - g_{il}\Gamma^l_{kj} - g_{lj}\Gamma^l_{ki} = 0$$

we choose $i = k = O$ and $j = \Lambda$, we obtain $\Gamma^O_{O\Lambda}(\mathcal{L}) \stackrel{*}{=} \Gamma^\Lambda_{OO}(\mathcal{L})$. These connection coefficients are determined by the acceleration four vector

$$A^i = \frac{dV^i}{d\tau} + \Gamma^i_{jk} V^j V^k.$$

In $\mathcal{K}_{\mathcal{L}}$ the spacelike components of this equation reduces to $A^\Lambda = c^2\Gamma^\Lambda_{OO}$. For example, in a laboratory on the Earth we have, neglecting Earth's rotation and revolution, $\Gamma^\Lambda_{OO} = g/c^2$.

Having determined the connection coefficients on $\mathcal{L}$ we can use them to write down the timelike and spacelike componenst of the derivative $\dfrac{{}^*D\mathbf{U}}{d\tau}$ separately:

$$\frac{{}^*DU^O}{d\tau} \stackrel{*}{=} \frac{DU^O}{d\tau} - \Gamma^O_{O\Lambda} V^O U^\Lambda \stackrel{*}{=}$$
$$\stackrel{*}{=} \frac{DU^O}{d\tau} - \frac{1}{c^2} A^\Lambda V^O U^\Lambda \stackrel{*}{=} \frac{DU^O}{d\tau} + \frac{1}{c^2}(\mathbf{A}\cdot\mathbf{U})V^O$$

(relation $A^\Lambda = -A_\Lambda$ has been used).

$$\frac{{}^*DU^\Lambda}{d\tau} \stackrel{*}{=} \frac{DU^\Lambda}{d\tau} - \Gamma^\Lambda_{OO} V^O U^O \stackrel{*}{=}$$
$$\stackrel{*}{=} \frac{DU^\Lambda}{d\tau} - \frac{1}{c^2} A^\Lambda V^O U^O \stackrel{*}{=} \frac{DU^\Lambda}{d\tau} - \frac{1}{c^2}(\mathbf{V}\cdot\mathbf{U})A^\Lambda.$$

These two equation can be united in the covariant form of the *Fermi-Walker derivative*:

$$\frac{{}^*D\mathbf{U}}{d\tau} = \frac{D\mathbf{U}}{d\tau} + \frac{1}{c^2}\left[(\mathbf{A}\cdot\mathbf{U})\mathbf{V} - (\mathbf{V}\cdot\mathbf{U})\mathbf{A}\right].$$

Appendix 4: The Papapetrou-equation.

Let us write (4.3) in the following form:

$$\frac{{}^{*}DU^i}{d\tau} = \frac{dU^i}{d\tau} + {}^{*}\Gamma^i_{kj} V^k U^j = 0, \tag{A}$$

where

$${}^{*}\Gamma^i_{kj} = \Gamma^i_{kj} - \frac{1}{c^2}(g_{kj}A^i - A_j\delta^i_k). \tag{B}$$

Spin can be represented either by the four-vector S^i or by the antisymmetric tensor S^{ij} which obey the constraint $V_iS^i = V_iS^{ij} = 0$ and are connected by the relations

$$S^{ij} = \frac{1}{c}\varepsilon^{ijkl}V_kS_l, \qquad S_i = -\frac{1}{c}\varepsilon_{ijkl}V^jS^{kl}.$$

Now the Fermi-Walker derivative of the first of them is

$$\frac{{}^{*}DS^{ij}}{d\tau} = \frac{1}{c}\varepsilon^{ijkl}V_k\frac{{}^{*}DS^i}{d\tau}$$

because the Fermi-Walker derivative of both the velocity and the weight 1 Levi-Civita pseudotensor is zero. This follows from the fact that in Fermi-Walker coordinates the Fermi-Walker derivative coincides with the ordinary derivative with respect to τ and in this system both quantites have constant components on $\mathcal{L}$. As a consequence of (A), therefore

$$\frac{{}^{*}DS^{ij}}{d\tau} = 0.$$

Since the Fermi-Walker derivation is a *derivation* it obeys the Leibniz-rule. Hence the detailed form of this equation is

$$\frac{dS^{ij}}{d\tau} + {}^{*}\Gamma^i_{kl}V^kS^{lj} + {}^{*}\Gamma^j_{kl}V^kS^{il} = 0.$$

The coefficients ${}^{*}\Gamma^i_{jk}$ are in general not symmetric in the lower pair of indices. Let us substitute (B):

$$\frac{DS^{ij}}{d\tau} + \frac{1}{c^2}V^iA_lS^{lj} - \frac{1}{c^2}V^jA_lS^{li} = 0.$$

Eliminating accelerations with the help of the transformation

$$A_iS^{ij} = \frac{DV_i}{d\tau}S^{ij} = -V_i\frac{DS^{ij}}{d\tau} = V_i\frac{DS^{ji}}{d\tau}$$

and we arrive at the Papapetrou-equation[22]

$$\frac{DS^{ij}}{d\tau} + \frac{1}{c^2}V^iV_k\frac{DS^{jk}}{d\tau} - \frac{1}{c^2}V^jV_k\frac{DS^{ik}}{dt} = 0. \tag{C}$$

[22] A. Papapetrou, Proc. Roy. Soc. **A209**, 248 (1951).

Appendix 5: Derivation of the equation (4.6).

The proof is conveniently performed in Schwarzschild coordinates rotating with angular velocity ω whose metric is given by the formula

$$\begin{aligned} ds^2 &= \left[(1 - r_g/r) - \left(\frac{r\omega}{c}\right)^2 \cdot \sin^2\vartheta\right] c^2dt^2 - 2\omega r^2 \sin^2\vartheta\, dt\, d\varphi - \\ &- \frac{dr^2}{1 - r_g/r} - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) = \\ &= \mathcal{A}c^2dt^2 - 2\omega r^2 \sin^2\vartheta\, dt\, d\varphi - \mathcal{B}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2). \end{aligned} \tag{A}$$

In this coordinate system the trajectory of a mass point revolving with an angular velocity ω in the plane $\vartheta = 90°$ becomes a straight line parallel to the time axis which intersects the coordinate plane $t = 0$, $\vartheta = 90°$ at the value of the radial coordinate r on a ray of arbitrary φ.

The Christoffel symbols in the rotating Schwarzschild coordinates are of three types. To the first group belong those coefficients which are common with the non-rotating Schwarzschild metric. They are independent of ω:

$$\begin{gathered} \Gamma^t_{tr} = -\Gamma^r_{rr} = \frac{r_g}{2r(r - r_g)} \qquad \Gamma^r_{\vartheta\vartheta} = -(r - r_g) \qquad \Gamma^r_{\varphi\varphi} = -(r - r_g)\sin^2\vartheta \\ \Gamma^\vartheta_{r\vartheta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r} \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta \cdot \cos\vartheta \qquad \Gamma^\varphi_{\vartheta\varphi} = \operatorname{ctg}\vartheta. \end{gathered} \tag{B}$$

The coefficients of the second group are proportional to ω:

$$\begin{gathered} \Gamma^\varphi_{tr} = \omega(\Gamma^\varphi_{\varphi r} - \Gamma^t_{tr}), \qquad \Gamma^r_{t\varphi} = \omega\Gamma^r_{\varphi\varphi}, \qquad \Gamma^\vartheta_{t\varphi} = \omega\Gamma^\vartheta_{\varphi\varphi}, \\ \Gamma^\varphi_{t\vartheta} = \omega\Gamma^\varphi_{\varphi\vartheta}, \qquad \Gamma^\vartheta_{tt} = \omega^2\Gamma^\vartheta_{\varphi\varphi}. \end{gathered} \tag{C}$$

The only remaining nonzero connection coefficient is

$$\Gamma^r_{tt} = (r - r_g)\left(\frac{c^2 r_g}{2r^3} - \omega^2\sin^2\vartheta\right) = (r - r_g)\left(\frac{MG}{r^3} - \omega^2\sin^2\vartheta\right). \tag{D}$$

In this metric the trajectory is[23]

$$\mathcal{L}: \qquad r = constant, \qquad \vartheta = \pi/2, \qquad \varphi = constant. \tag{E}$$

On this trajectory the proper time and the coordinate time are proportional to each other:

$$d\tau = \sqrt{g_{tt}dt^2 - \frac{1}{c^2}g_{\varphi\varphi}d\varphi^2} = \sqrt{1 - r/r_g - (r\omega/c)^2}\, dt.$$

[23] In the rotating Schwarzschild coordinate system the geodetic equation $\mathbf{A} = 0$ is fulfilled if $\Gamma^r_{tt} = 0$ i.e. when $MG/r^2 = r\omega^2$ ($\vartheta = 90°$). In classical gravity this same relation expresses the equality of the gravitational and centrifugal forces in rotating coordinate system.

Hence, in (4.5) the independent variable τ may be replaced by the coordinate time t. As a result, the formula of the tangent vector to $\mathcal{L}$ becomes very simple:

$$\mathbf{V} = (V^t,\, V^r,\, V^\vartheta,\, V^\varphi) = (1,\, 0,\, 0,\, 0).$$

The covariant components of $\mathbf{V}$ are $(V_t,\, V_r,\, V_\vartheta,\, V_\varphi) = (g_{tt},\, 0,\, 0,\, g_{t\varphi})$ and from $\mathbf{V} \cdot \left(\frac{D\mathbf{S}}{d\tau}\right)_\perp = 0$ we have

$$\left(\frac{D\mathbf{S}}{d\tau}\right)_\perp \sim \left(-g_{t\varphi}\frac{DS^\varphi}{dt},\, \frac{DS^r}{dt},\, 0,\, g_{tt}\frac{DS^\varphi}{dt}\right),$$

where $\sim$ means proportionality. When the only nonzero component of $\frac{D\mathbf{S}}{d\tau}$ is $\frac{DS^t}{dt}$ then the transverse part is equal to zero as should be.

Equations (4.5), therefore, become

$$\frac{dS^r}{dt} + \Gamma^r_{tt}S^t + \Gamma^r_{t\varphi}S^\varphi = 0 \tag{F}$$

$$\frac{dS^\varphi}{dt} + \Gamma^\varphi_{tr}S^r = 0 \tag{G}$$

$$\mathbf{S}\cdot\mathbf{V} = g_{tt}S^t + g_{t\varphi}S^\varphi = \left[1 - \frac{r_g}{r} - \left(\frac{r\omega}{c}\right)^2\right] S^t - \omega r^2 S^\varphi = 0.$$

From the last line we express S^t and substitute into (F). Then we obtain a homogeneous pair of equations for S^r and S^φ. Excluding the latter we arrive at the equation

$$\frac{d^2S^r}{dt^2} + \Omega^2 S^r = 0,$$

in which

$$\Omega^2 = \left(\frac{g_{t\varphi}\Gamma^r_{tt} - g_{tt}\Gamma^r_{t\varphi}}{g_{tt}}\right)\Gamma^\varphi_{tr}.$$

All quantities on the r. h. s. have been specified above. Substituting them we obtain

$$\Omega^2 = \omega^2 \frac{\left(1 - \frac{3}{2}\frac{r_g}{r}\right)^2}{1 - \frac{r_g}{r} - \left(\frac{\omega r}{c}\right)^2},$$

from which (4.6) follows.