The Rotating Magnet

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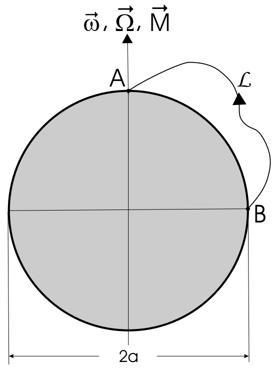
Abstract: Axisymmetric permanent magnets become electrically polarized due to their rotation around the symmetry axis. This phenomenon is considered in detail for both conducting and dielectric magnets. The results are applied to the Earth which is predicted to be electrically polarized. It is suggested that this polarization can be detected in a tethered satellite experiment.

The World Dynamo

Many years ago I read as a schoolboy an exciting book about a dogged ingeneer who took it into his head to lay down a massive electric cable along a large segment of a meridian, for, according to his calculations, the rotation of the Earth in its own magnetic field should induce in the cable currents of enormous strength and this "world dynamo" — that was the name of the book — would supply mankind with cheep electricity.

Though I was charmed with the idea I had something on my mind: What if the magnetic field rotated together with the Earth? If it did the cable would never cross the lines of force of the field and no current would be induced. I did not realized until much later that my question itself was rather problematic since the meaning of rotation of an axisymmetric magnetic field around its symmetry axis was far from being obvious. If lines of force existed in reality and the motion of either of them could be followed in time, my question would be all right and could in principle be answered. But these lines are only mathematical abstractions deviced to aid the visualization of the field structure and have no real existence. The magnetic field of a magnet is in fact independent of whether the magnet rotates around its symmetry axis or not — in this respect the engineer was certainly right.

In what follows we will consider a homogeneous spherical magnet of radius a, rotating with a constant angular velocity $\boldsymbol{\Omega}$, whose magnetization density $\boldsymbol{M}$ is parallel to the axis of rotation (assumed to be the z-axis). The cable of the world dynamo will be represented by a rigid linear conductor $\mathcal{L}$, not necessarily a plane curve, which connects the "north pole" and the "equator" of the sphere. The conductor $\mathcal{L}$ may also rotate with an angular velocity $\boldsymbol{\omega}$ which is parallel to $\boldsymbol{\Omega}$ but may differ from it in magnitude. (In the world dynamo we have $\boldsymbol{\omega} = \boldsymbol{\Omega}$ and $\mathcal{L}$ lies along a meridian on the surface of the Earth but it will turn out expedient to deal with the more general case.) At the endpoints A ($\vartheta = 0°$) and B ($\vartheta = 90°$) the linear conductor is connected electrically to the magnet by means of sliding contacts so as to make $\mathcal{L}$ part of an electric circuit closed through the magnet.



Below we will restrict ourselves to the discussion of the physical basis, underlying the world dynamo idea. It will be left to the reader to judge whether such an extraordinary power plant if realized in practice would indeed be continuously supplying electric power or not.

The Rotating Conducting Magnet

Consider a rotating metallic magnet of conductivity γ temporarily stripped of the linear conductor $\mathcal{L}$. In a conductor at rest the connection between the current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$ is given by the Ohm's law $\boldsymbol{J} = \gamma\boldsymbol{E}$. When the conductor is moving the electric field must be supplemented by the electromotive force ($\boldsymbol{V} \times \boldsymbol{B}$) and in this more general

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case the Ohm's law becomes

$$\boldsymbol{J} = \gamma[\boldsymbol{E} + (\boldsymbol{V} \times \boldsymbol{B})]. \tag{1}$$

Let us choose the origin of the coordinate system at the center of the sphere. Then the element of the magnet at $\boldsymbol{r}$ will have the velocity

$$\boldsymbol{V} = (\boldsymbol{\Omega} \times \boldsymbol{r}). \tag{2}$$

From this formula it is obvious that we are working in the *inertial system* in which the center of the magnet is at rest rather than in the system, rotating together with the magnet around this point. In what follows we will never replace our reference frame with the corotating one.

Just as it is in the case of a conductor at rest the current density in a rotating conducting sphere also vanishes under stationary conditions. In the latter case, however, the electric field does not disappear together with the current density since when $\boldsymbol{J} = 0$ we obtain from (1) the electric field

$$\boldsymbol{E} = -(\boldsymbol{V} \times \boldsymbol{B}) \qquad (r < a) \tag{3}$$

which is associated with some definite volume and surface charge densities ρ and σ. Eq. (3) and the Maxwell-equation $\operatorname{div} \boldsymbol{E} = \rho/\epsilon_0$ determine the volume charge density:

$$\rho = -\epsilon_0 \operatorname{div}(\boldsymbol{V} \times \boldsymbol{B}).$$

As it is known from magnetostatics the induction within a homogeneously magnetized sphere is equal to

$$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M} \qquad (r < a). \tag{4}$$

Therefore,

$$(\boldsymbol{V} \times \boldsymbol{B}) = \frac{2}{3}\mu_0 (\boldsymbol{V} \times \boldsymbol{M})$$

and, using (2), we obtain

$$\begin{aligned}(\boldsymbol{V} \times \boldsymbol{M}) &= \big((\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{M}\big) = \\ &= (\boldsymbol{M} \cdot \boldsymbol{\Omega})\boldsymbol{r} - (\boldsymbol{M} \cdot \boldsymbol{r})\boldsymbol{\Omega}.\end{aligned} \tag{5}$$

Let us take now into account that the constant vectors $\boldsymbol{M}$ and $\boldsymbol{\Omega}$ are parallel to each other, $\operatorname{div} \boldsymbol{r} = 3$ and, finally,

$$\operatorname{div}\big((\boldsymbol{M} \cdot \boldsymbol{r})\boldsymbol{\Omega}\big) = M\Omega.$$

Then

$$\operatorname{div}(\boldsymbol{V} \times \boldsymbol{M}) = 2M\Omega \tag{6}$$

and

$$\rho = -\frac{4}{3}\epsilon_0 \mu_0 M \Omega. \tag{7}$$

Using (4) it is easy to show that for the electric field (3) $\operatorname{rot} \boldsymbol{E} = 0$ which is the second Maxwell-equation for $\boldsymbol{E}$ when the fields are constant in time.

The surface charge density can be calculated as in electrostatics. We have

$$\sigma = \epsilon_0 (E_r^+ - E_r^-), \qquad (r = a) \tag{8}$$

where E_r^+ and E_r^- are the radial components of the electric field on the outer (+) and inner (–) side of the surface of the magnet. Our previous formulae permit us to write

$$E_r^- = -(\boldsymbol{V} \times \boldsymbol{B})_r = -V_\vartheta B_\varphi + V_\varphi B_\vartheta. \tag{9}$$

Since $\boldsymbol{B}$ has only a z-component we have

$$B_\varphi = 0 \qquad \text{and} \qquad B_\vartheta = -\frac{2}{3}\mu_0 M \sin\vartheta. \tag{10}$$

The only nonzero component of $\boldsymbol{V}$ is V_φ which is equal to $\Omega r \sin\vartheta$. Hence

$$E_r^- = -\frac{2}{3} a \mu_0 M \Omega \sin^2\vartheta \qquad (r = a). \tag{11}$$

In order to calculate E_r^+ the electrostatic potential Φ outside the sphere must be known. From the potential the electric field is obtained as a gradient:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi. \tag{12}$$

Outside the sphere the charge density is zero and Φ obeys the Laplace-equation

$$\triangle \Phi = 0, \qquad (r > a) \tag{13}$$

from the solution of which E_r^+ can be calculated as

$$E_r^+ = -\left.\frac{\partial \Phi}{\partial r}\right|_{r=a}. \tag{14}$$

In Appendix 1 we show that

$$\Phi = -\frac{1}{9}a^5\mu_0 M\Omega \cdot \frac{1}{r^3}(3\cos^2\vartheta - 1) \qquad (r \geq a), \tag{15}$$

from which we obtain

$$E_r^+ = -\frac{1}{3}a\mu_0 M\Omega(3\cos^2\vartheta - 1). \tag{16}$$

Therefore, the surface charge density is given by the formula

$$\sigma = \frac{1}{3}a\epsilon_0\mu_0 M\Omega(1 + \cos^2\vartheta). \tag{17}$$

It is straightforward to show that the total surface charge compensates exactly the total volume charge.

Since both the induced volume and surface charges rotate together with the magnet they give rise to corresponding current densities. In the case of the volume charge density this current density is equal to

$$\Delta\boldsymbol{J} = \rho\boldsymbol{V} = -\frac{4}{3}\epsilon_0\mu_0 M\Omega\boldsymbol{V}. \tag{18}$$

This $\Delta\boldsymbol{J}$ does not contribute to the l.h.s. of the Ohm's law (1) because it arises from the rotation rather from the effect of the field $\boldsymbol{E}$. However, $\Delta\boldsymbol{J}$ generates, through the Maxwell-equation $\mathrm{rot}\,\Delta\boldsymbol{B} = \mu_0\Delta\boldsymbol{J}$, the magnetic field $\Delta\boldsymbol{B}$ which must be added to $\boldsymbol{B}$ on the r.h.s. of (1). As a consequence this same correction appears on the r.h.s. of (3) also but, as we now show, gives only vanishingly small contribution to $\boldsymbol{E}$.

Dimensional considerations based on both physical and geometrical dimensions lead to the solution $|\Delta\boldsymbol{B}| \sim a\mu_0|\Delta\boldsymbol{J}|$ ($\sim$ denotes "order of magnitude equality"). Hence, taking into account (18) and (4), we have

$$\left|\frac{\Delta\boldsymbol{B}}{\boldsymbol{B}}\right| \sim a\mu_0\frac{\epsilon_0\mu_0 M\Omega V}{\mu_0 M} = \frac{a\Omega V}{c^2} \sim \frac{V^2}{c^2}.$$

This correction is indeed very small for both laboratory magnets and celestial bodies and so will be neglected. The factors $(1 - V^2/c^2)^{1/2}$ which should be included at certain places into our formulae are left out of consideration for the same reason.

The Rigidly Fixed Contour

Let us assume now that the contour $\mathcal{L}$ is rigidly fixed to the magnet ($\boldsymbol{\omega} = \boldsymbol{\Omega}$). If it did not rotate the electromotive force in it would be given by the formula

$$\mathcal{E} = \int_{\mathcal{L}} \boldsymbol{E}\cdot d\boldsymbol{l}.$$

For a rotating contour an electromotive force induced by the motion also contributes to $\mathcal{E}$:

$$\mathcal{E} = \int_{\mathcal{L}} \left[\boldsymbol{E} + (\boldsymbol{V}\times\boldsymbol{B})\right]\cdot d\boldsymbol{l}. \tag{19}$$

When calculating the integral the contour must be assumed fixed in our coordinate system since its motion is already taken into account by the second term of the integrand[1].

In (19) the electromotive force is a sum of an electric component $\mathcal{E}_e = \int_{\mathcal{L}} \boldsymbol{E}\cdot d\boldsymbol{l}$ and a magnetic (or motional) component $\mathcal{E}_m = \int_{\mathcal{L}} (\boldsymbol{V}\times\boldsymbol{B})\cdot d\boldsymbol{l}$. Since the former is originated from the electric polarization of the rotating magnet it is equal to the potential difference

$$\mathcal{E}_e = \Phi_B - \Phi_A, \tag{20}$$

which, for fixed endpoints, is independent of the form of the contour.

But $\mathcal{E}_m$ is contour independent as well since for a closed contour

$$\oint(\boldsymbol{V}\times\boldsymbol{B})\cdot d\boldsymbol{l} = 0. \tag{21}$$

This is the consequence of the Stokes-theorem

$$\oint(\boldsymbol{V}\times\boldsymbol{B})\cdot d\boldsymbol{l} = \int_{\Sigma} \mathrm{rot}(\boldsymbol{V}\times\boldsymbol{B})\cdot\boldsymbol{n}\,d\Sigma,$$

in which Σ is any surface bounded by the closed contour and $\boldsymbol{n}$ is its normal vector. In Appendix 2 we will prove that at any point of space (i.e. both inside and outside the sphere) the equality

$$\mathrm{rot}(\boldsymbol{V}\times\boldsymbol{B}) = 0 \tag{22}$$

[1]This method of calculation is justified if the displacement of the contour is negligible during the time interval the electromagnetic signal passes through it. Problems which we are interested in do not require higher accuracy.

holds from which (21) and the contour independence of $\mathcal{E}_m$ follow (for fixed endpoints).

Since both $\mathcal{E}_e$ and $\mathcal{E}_m$ are contour independent the same is true for the total electromotive force $\mathcal{E}$ too, therefore, the contour in (19) may be chosen for convenience. The best choice is to direct it entirely within the magnet since, according to (3), along such a contour the integrand of (19) vanishes. Hence we conclude that in any linear contour which rotates together with the magnet and whose endpoints lie on the surface no electromotive force is induced. As it follows from Appendix 2 this conclusion remains valid also for an axisymmetric conducting magnet of any form, rotating around its symmetry axis.

The Unipolar Induction

When $\mathcal{L}$ rotates with respect to the magnet ($\boldsymbol{\omega} \neq \boldsymbol{\Omega}$) the electromotive force in it consists of the same kind of terms as in the corotating contour. For $\mathcal{E}_e$ (15) remains valid. The potentials Φ_A and Φ_B can be calculated from (15). In A and B we have $\vartheta = 0°$ and $90°$ respectively and in both cases $r = a$, therefore

$$\mathcal{E}_e = \Phi_B - \Phi_A = \frac{1}{3}a^2\mu_0 M\Omega. \tag{23}$$

As we saw in the preceding section for a corotating contour $\mathcal{E}_m = -\mathcal{E}_e$. This electromotive force depends on the velocity of the points of the contour in the reference frame chosen and in the case of corotation it is proportional to Ω. Then, for a contour, rotating independently of the magnet, $\mathcal{E}_m$ coincides with the *negative* of (23) in which Ω is replaced by ω:

$$\mathcal{E}_m = -\frac{1}{3}a^2\mu_0 M\omega. \tag{24}$$

Therefore, the full electromotive force is given by the equation

$$\mathcal{E} = \frac{1}{3}a^2\mu_0 M(\Omega - \omega). \tag{25}$$

Owing to the sliding contacts, $\mathcal{L}$ closes through the magnet and since the latter's conductivity is different from zero a current will flow in $\mathcal{L}$.

According to (25) electromotive force and current arise even in a contour at rest ($\boldsymbol{\omega} = 0$). This phenomenon known as the *unipolar induction* is rather paradoxical since the current can explain neither by the law of induction (since the magnetic field is constant in time) nor as a motional induction (since the contour is at rest). Unipolar induction originates solely from the electric polarization of the magnet. For a spherical magnet its magnitude can be calculated from (23) but the sphericity is, of course, not essential for the phenomenon to occur. In a Faraday-disk magnetized along its axis of rotation current will be generated even in the absence of an external magnetic field.

As it is seen from (25) the electromotive force depends on the *relative* rotation of the magnet and the contour. This is quite an unexpected result since rotation is *absolute*: A deformable sphere, rotating in an inertial frame takes on the shape of an ellipsoid of rotation. This deformation is the manifestation of the absolute rotation since it exists irrespective of the frame of reference from which the sphere is observed.

The role of rotation in electrodynamics is by no means different. The electric polarization of the rotating magnet is an objective (absolute) phenomenon in the same sense as the deformation of a rotating sphere since it demonstrates unequivocally that it is the magnet — and not the contour — which is rotating. Curiously enough, *in the special case of the electromotive force in* $\mathcal{L}$ it is only the relative rotation which counts. But this is so only when the magnet conducts electricity. For a magnet made of insulator relation (25) ceases to be valid and $\mathcal{E}$ turns out to depend on the angular velocities separately rather than on their difference. This question will be studied in the next section.

The Rotating Dielectric Magnet

Assume now that our magnet does not conduct electricity ($\gamma = 0$) but, instead, electrically polarizable ($\epsilon \geq \epsilon_0$). Then, under stationary conditions, the l.h.s. of (1) is obviously equal to zero but since now $\gamma = 0$ Eq. (3) does not follow from this fact.

The field $\boldsymbol{E} + (\boldsymbol{V} \times \boldsymbol{B})$ which in a moving conductor determines the current through the Ohm's law makes a dielectric polarized:

$$\boldsymbol{P} = \chi\epsilon_0\big[\boldsymbol{E} + (\boldsymbol{V} \times \boldsymbol{B})\big] = \boldsymbol{P}^{(1)} + \boldsymbol{P}^{(2)} \tag{26}$$

(χ is the dielectric susceptibility). In the above equation $\boldsymbol{P}^{(1)} = \chi\epsilon_0\boldsymbol{E}$ is the electrostatic while $\boldsymbol{P}^{(2)} = \chi\epsilon_0(\boldsymbol{V}\times\boldsymbol{B})$ is the magnetically induced (or motional) polarization.

This is, however, not yet the full polarization since there is a third contribution

$$\boldsymbol{P}^{(3)} = \epsilon_0\mu_0(\boldsymbol{V}\times\boldsymbol{M}) \tag{27}$$

predicted by relativity theory, according to which elementary magnetic dipoles $\boldsymbol{m}$ of a moving permanent magnet acquire electric dipole moment equal to $\epsilon_0\mu_0(\boldsymbol{V}\times\boldsymbol{m})$.

The origin of this phenomenon may be understood directly from the equivalence of the inertial frames of reference without resort to the apparatus of relativity theory.

Consider an elementary magnetic dipole $\boldsymbol{m}$ which is at rest in an inertial frame of reference and an elementary linear conductor $\boldsymbol{dl}$ which is moving with constant velocity $\boldsymbol{v}$. The electromotive force $d\mathcal{E}$ induced in this elementary conductor by its motion is equal to $d\mathcal{E} = (\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{dl}$ in which $\boldsymbol{B}$ is the field of the dipole at the position of the conductor. The absence of any absolute frame of reference requires that when the conductor is at rest and the dipole is moving with constant velocity $-\boldsymbol{v}$ the electromotive force induced remain the same as before.

The magnetic field of the moving dipole at the fixed position of $\boldsymbol{dl}$ varies in time and, therefore, it brings about, through Maxwell-equations, an electric field $\boldsymbol{E}$ which in turn produces an electromotive force $\boldsymbol{E}\cdot\boldsymbol{dl}$. Simple calculation shows that, contrary to the expectation, $\boldsymbol{E}\cdot\boldsymbol{dl} \neq d\mathcal{E}$. Equality is obtained only if in $\boldsymbol{E}$ one takes into account the electric field of the electric dipole moment $\epsilon_0\mu_0\big((-\boldsymbol{v})\times\boldsymbol{m}\big)$ acquired by $\boldsymbol{m}$ due to its velocity $-\boldsymbol{v}$.

The volume charge densities produced by all three types of polarization are given by the equation

$$\rho_i = -\operatorname{div}\boldsymbol{P}^{(i)} \qquad (i = 1,2,3). \tag{28}$$

Through Gauss-theorem this formula determines the surface charge densities as

$$\sigma_i = P_r^{(i)} \qquad (r = a, \quad i = 1,2,3). \tag{29}$$

In words: The surface charge densities are given by the normal component of the polarization vectors on the inner side of the surface.

According to the first Maxwell-equation

$$\operatorname{div}\epsilon_0\boldsymbol{E} = \rho_1 + \rho_2 + \rho_3.$$

If we introduce the induction vector by the formula

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}^{(1)}$$

the above equation takes on the form

$$\operatorname{div}\boldsymbol{D} = \rho_2 + \rho_3 \tag{30}$$

and we arrive at a standard electrostatic problem: Consider a sphere of constant dielectric permeability ϵ. Calculate the electrostatic potential for given volume and surface charge densities $(\rho_2+\rho_3)$ and $(\sigma_2+\sigma_3)$.

Eq. (4) and the relation $\chi\epsilon_0 = (\epsilon - \epsilon_0)$ permit us to write

$$\boldsymbol{P}^{(2)} + \boldsymbol{P}^{(3)} = \frac{1}{3}(2\epsilon+\epsilon_0)(\boldsymbol{V}\times\mu_0\boldsymbol{M})$$

which in turn leads through (6) to

$$\begin{aligned}\rho_2+\rho_3 &= -\operatorname{div}(\boldsymbol{P}^{(2)}+\boldsymbol{P}^{(3)}) = \\ &= -\frac{2}{3}(2\epsilon+\epsilon_0)\mu_0 M\Omega.\end{aligned} \tag{31}$$

This expression will be substituted into the r.h.s. of (30).

Similarly, we obtain for the surface density the equation

$$\sigma_2+\sigma_3 = P_r^{(2)} + P_r^{(3)} = \frac{1}{3}(2\epsilon+\epsilon_0)(\boldsymbol{V}\times\mu_0\boldsymbol{M})_r.$$

Using (5), the relation $(\boldsymbol{r}\cdot\boldsymbol{M}) = rM\cos\vartheta$ and $\Omega_r = \Omega\cos\vartheta$ we have

$$\sigma_2+\sigma_3 = \frac{1}{3}(2\epsilon+\epsilon_0)a\mu_0 M\Omega\sin^2\vartheta. \tag{32}$$

We write in (30) $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and through $E = -\boldsymbol{\nabla}\Phi$ introduce the potential Φ again. Then

$$\triangle\Phi = \begin{cases} -\dfrac{1}{\epsilon}(\rho_2+\rho_3) & (r<a) \\ 0 & (r>a). \end{cases} \tag{33}$$

The boundary condition for $\boldsymbol{D}$ is fixed by the surface densities as

$$D_r^+ - D_r^- = \sigma_2 + \sigma_3,$$

in which $\boldsymbol{D}^+$ and $\boldsymbol{D}^-$ are the inductions on the outer and inner sides of the surface. This condition expressed through the potential becomes

$$-\epsilon_0 \frac{\partial \Phi_+}{\partial r} + \epsilon \frac{\partial \Phi_-}{\partial r} = \sigma_2 + \sigma_3 \qquad (r = a) \tag{34}$$

(Φ_+ and Φ_- are the potentials outside and inside the sphere).

Eq. (33) will be solved in Appendix 3 with the result

$$\Phi = -\frac{2\epsilon + \epsilon_0}{9(2\epsilon + 3\epsilon_0)} \mu_0 M \Omega \left(3\cos^2\vartheta - 1\right) \times \\ \times \begin{cases} r^2 & (r < a) \\ \dfrac{a^5}{r^3} & (r > a), \end{cases} \tag{35}$$

From this we obtain for the electric part of the electromotive force $\mathcal{E}$ the expression

$$\mathcal{E}_e = \Phi_B - \Phi_A = \frac{2\epsilon + \epsilon_0}{3(2\epsilon + 3\epsilon_0)} a^2 \mu_0 M \Omega. \tag{36}$$

The magnetic part is still given by (24). Hence, the full electromotive force is given as

$$\mathcal{E} = \frac{1}{3} a^2 \mu_0 M \left(\frac{2\epsilon + \epsilon_0}{2\epsilon + 3\epsilon_0} \Omega - \omega \right). \tag{37}$$

As we have already mentioned in this case it is *not* the relative rotation which determines $\mathcal{E}$. In spite of the existence of this electromotive force no stationary current will flow through $\mathcal{L}$ since the magnet's conductivity is zero but the potential difference between the points A and B may be observed.

Since conductors are "infinitely easily" polarizable substances (37) must be reduced in the limit $\epsilon \longrightarrow \infty$ to (25) which is indeed the case.

The numerator of (36) can be written in the form $\left[2(\epsilon - \epsilon_0) + 3\epsilon_0\right]$ in which the term $3\epsilon_0$ derives from $\boldsymbol{P}^{(3)}$. Since this term does not contain ϵ it drops out of the limit $\epsilon \longrightarrow \infty$. Does this mean that the elementary magnets in a conducting permanent magnet do not acquire electric dipole moment due to their motion?

Of course, not. According to (3), in a conducting magnet the rotation determines the electric field directly, independently of whether the latter is produced by polarization charges or motion-induced electric dipole moments. In a dielectric magnet, on the contrary, it is the polarization rather than the field itself which is fixed by the rotation and so in this case it is crucial to take into account all possible types of polarizations.

When the magnet is neither conducting ($\gamma = 0$) nor polarizable ($\epsilon = \epsilon_0$) the electromotive force in $\mathcal{L}$ is still different from zero:

$$\mathcal{E} = \frac{1}{3} A^2 \mu_0 M \left(\frac{3}{5} \Omega - \omega \right).$$

This electromotive force originates solely from $\boldsymbol{P}^{(3)}$.

The Tethered Satellite

In march of 1996 a spherical 1.6-meter diameter satellite was released out into space from the payload bay of Space Shuttle Columbia during its orbiting at a height about 90 km above the Earth. Its tether, a long conducting cable, served (among others) to generate electric power due to the electromotive force $\mathcal{E}_m$ induced in it by the Erth's magnetic field. Free electrons in the thin ionosphere where the Space Shuttle operated were attracted to the satellite. The electrons travelled along the tether to the orbiter. The electric circuit was closed by means of an electron generator on the orbiter which returned charged particles back into the ionosphere.

The electromotive force $\mathcal{E}_m$ is the greater the longer the tether is. The latter was a 21-kilometer-long leash but it broke when the satellite was extended 19.7 kilometers. The experiment, however, could not be considered as a failure. Up to the time of the severing of the tether, the orbiter-tether-satellite system had been generating 3,500 volts and up to 0.5 amps of current.

Considerations of the preceding sections suggest that, since the Earth rotates in its own magnetic field, it must be electrically polarized and the electric field of the polarization charges must give rise to an electromotive force $\mathcal{E}_e$ in the tether which also contributes to the current in it. To have an order of magnitude estimate we assume that (1) the Earth core is a homogeneous permanent magnet of nonzero conductivity γ, (2) the rotational and magnetic axes of the Earth coincide and (3) the mantle's and the atmosphere's polarizabilities are negligible. Neither of these assumptions is correct but, perhaps, they provide an acceptable starting point.

Consider a tethered orbiter-satellite system, orbiting above the Equator (xy plane) on a circular orbit of radius r. Assume further that the tether, a linear conductor of length $\Delta l \ll r$ is oriented along the radius and the positive direction on it points toward the increase of r. Then

$$\Delta\mathcal{E}_m = (\boldsymbol{v} \times \boldsymbol{B})_r \, \Delta l.$$

$\boldsymbol{v}$ is the orbiting velocity with respect to the (practically inertial) frame of reference with its origin in the center of the Earth and orientation defined by the fixed starts. The magnitude of $\boldsymbol{v}$ is equal to ωr where ω is the angular velocity of the orbiter. In the orbital plane the Earth's magnetic field $\boldsymbol{B}$ has only z-component equal to

$$+\frac{1}{3}\mu_0 M \left(\frac{a_c}{r}\right)^3$$

where a_c is the radius of the Earth core. Hence

$$\Delta\mathcal{E}_m = +\frac{1}{3}\mu_0 M \omega \frac{a_c^3}{r^2} \, \Delta l.$$

This positive $\Delta\mathcal{E}_m$ gives rise to an electron current directed toward the Earth and, therefore, the satellite has to be orbiting *above* the shuttle.

On the other hand,

$$\Delta\mathcal{E}_e = E_r \, \Delta l = -\frac{\partial\Phi}{\partial r} \, \Delta l.$$

The r.h.s. is to be calculated at $\vartheta = 90°$. According to (15)

$$\Delta\mathcal{E}_e = \frac{1}{3}\mu_0 M \Omega \frac{a_c^5}{r^4} \, \Delta l,$$

therefore,

$$\left|\frac{\Delta\mathcal{E}_e}{\Delta\mathcal{E}_m}\right| = \left(\frac{a_c}{r}\right)^2 \frac{\Omega}{\omega}$$

where Ω is the angular velocity of the Earth's rotation. Since $a_c/r \approx 1/2$ and $\Omega/\omega \approx 1/20$, $\Delta\mathcal{E}_e$ is less than $\Delta\mathcal{E}_m$ only by about two orders of magnitude.[2] Moreover, if the orbital plane is perpendicular to the Equator (i.e. it goes through the Poles and is, of course, at rest with respect to the fixed stars) then, since $\boldsymbol{v} \times \boldsymbol{B}$ is perpendicular to the radial direction, $\Delta\mathcal{E}_m = 0$ and it is $\Delta\mathcal{E}_e$ *alone* which contributes to the electric current in the tether. Though polarizability of the ionospher may substantially alter (or even invalidate) this conclusion the possibility of the electric polarization of the Earth seems worth of further consideration.

Appendix 1

We are looking for the solution of the equation (13) at $a > r$ which tends to zero faster than $1/r$ (the total charge of the sphere is zero) and the tangential component of the electric field on the outer side of the surface $r = a$ coincides with the tangential component of the field (3) inside the sphere. Owing to the surface charges which compensate the volume charge the normal component of $\boldsymbol{E}$ will be discontinuous at $r = a$.

Axial symmetry requires E_φ to vanish and so the tangential component is given by

$$E_\vartheta = -\frac{1}{a} \left.\frac{\partial\Phi}{\partial\vartheta}\right|_a$$

alone. According to (3) it is equal to

$$E_\vartheta = -(\boldsymbol{V} \times \boldsymbol{B})_\vartheta = -V_\varphi B_r + V_r B_\varphi.$$

Both factors of the second term are equal to zero while in the first term

$$V_\varphi = \Omega r \sin\vartheta, \qquad B_r = \frac{2}{3}\mu_0 M \cos\vartheta.$$

[2] Since charged particles move much faster than the tethered system, the effect of the Coulomb-force due to the Earth's polarization for them is negligible with respect to the Lorentz-force.

Hence

$$\frac{1}{a}\left.\frac{\partial\Phi}{\partial\vartheta}\right|_a = \frac{2}{3}\mu_0 M\Omega a\sin\vartheta\cos\vartheta. \qquad (38)$$

The axisymmetric solution of (13) which obeys all the requirements formulated is given by the equation

$$\Phi(r,\vartheta) = \frac{A}{r^3}P_2(\cos\vartheta) = \frac{A}{2r^3}(3\cos^2\vartheta - 1) \qquad (r \geq a). \qquad (39)$$

$P_2(\cos\vartheta)$ is the 2-nd Legendre-polynomial, A is a constant which is fixed by (38) as

$$A = -\frac{2}{9}a^5\mu_0 M\Omega$$

Substituting this into (39) we obtain (15).

Appendix 2

In the textbook formula

$$\mathrm{rot}(\boldsymbol{V}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{V} - (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{V}\,\mathrm{div}\,\boldsymbol{B} - \boldsymbol{B}\,\mathrm{div}\,\boldsymbol{V}$$

the last two terms vanish as a consequence of $\mathrm{div}\,\boldsymbol{B} = 0$ and $\mathrm{div}\,\boldsymbol{V} = 0$. In Cartesian-coordinates $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$ and we have

$$\mathrm{rot}_j(\boldsymbol{V}\times\boldsymbol{B}) = B_i\partial_i V_j - V_i\partial_i B_j \qquad (40)$$

where summation convention over repeated indices is understood (e.g. $B_i\partial_i = B_x\partial_x + B_y\partial_y + B_z\partial_z$).

Consider now Eq. (2) . For the components in Cartesian coordinates we have the formula $(\boldsymbol{\Omega}\times\boldsymbol{r})_i = \epsilon_{ijk}\Omega_j x_k$ in which $(x_1, x_2, x_3) \equiv (x, y, z)$, and ϵ_{ijk} is the fully antisymmetric unit tensor (ϵ-symbol)[3]. Using this in (40), we have

$$\mathrm{rot}_j(\boldsymbol{V}\times\boldsymbol{B}) = \epsilon_{jkl}B_i\partial_i(\Omega_k x_l) - \epsilon_{ikl}\Omega_k x_l\partial_i B_j.$$

Since the current desity is zero throughout, $\boldsymbol{B}$ is rotation-free and $\partial_i B_j = \partial_j B_i$. Hence

$$\mathrm{rot}_j(\boldsymbol{V}\times\boldsymbol{B}) = \epsilon_{jkl}B_i\partial_i(\Omega_k x_l) - \epsilon_{ikl}\Omega_k x_l\partial_j B_i.$$

In the first term

$$\partial_i(\Omega_k x_l) = \Omega_k\frac{\partial x_l}{\partial x_i} = \Omega\delta_{il}$$

[3] $\epsilon_{ijk} = +1$ or -1 depending on whether ijk is an even or odd permutation of 123. If some of the indices ijk coincide the value of the symbol is zero.

which is equal to Ω_k at $i = l$ and vanishes at $i \neq l$, therefore

$$\mathrm{rot}_j(\boldsymbol{V}\times\boldsymbol{B}) = \epsilon_{jki}B_i\Omega_k - \epsilon_{ikl}\Omega_k x_l\partial_j B_i.$$

In the second term we perform a transformation of the opposite sense: $x_l\partial_j B_i = \partial_j(x_l B_i) - \delta_{lj}B_i$. If we substitute this into the previous formula and use the constancy of Ω_k and the ϵ-symbol we obtain

$$\mathrm{rot}_j(\boldsymbol{V}\times\boldsymbol{B}) = -\partial_j(\epsilon_{ikl}B_i\Omega_k x_l) + (\epsilon_{jki} + \epsilon_{ikj})B_i\Omega_k.$$

The sum of the ϵ-symbols is zero and under the sign of partial derivation we recognize the mixed product of the vectors $\boldsymbol{B}, \boldsymbol{\Omega}$ and $\boldsymbol{r}$:

$$\mathrm{rot}_j(\boldsymbol{V}\times\boldsymbol{B}) = -\partial_j\big[\boldsymbol{B}\cdot(\boldsymbol{\Omega}\times\boldsymbol{r})\big].$$

The mixed product of three vectors is equal to the determinant formed from their Cartesian-components. Since the magnet is assumed axisymmetric the vector $\boldsymbol{B}$ at the point $\boldsymbol{r}$ lies in the plane defined by Ω (i.e. the z-axis) and the direction of $\boldsymbol{r}$. Hence the determinant vanishes and the field $(\boldsymbol{V}\times\boldsymbol{B})$ is indeed rotationless. Our proof of this fact is obviously valid for any axisymmetric magnet which rotates around its symmetry axis.

Appendix 3

We are seeking the solution of (34) which is everywhere finite and continuous (even at $r = a$). (32) suggests the expected ϑ dependence of the solution:

$$\Phi_- = Ar^2P_2(\cos\vartheta) + f(r) = \frac{A}{2}r^2(3\cos^2\vartheta - 1) + f(r)$$

$$\Phi_+ = \frac{B}{r^3}P_2(\cos\vartheta) = \frac{B}{2r^3}(3\cos^2\vartheta - 1)$$

in which A and B are constants and $f(r)$ is a particular solution of

$$\triangle f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = -\frac{1}{\epsilon}(\rho_2 + \rho_3) \qquad (41)$$

which will be chosen to vanish at $r = a$:

$$f(r) = \frac{\rho_2 + \rho_3}{6\epsilon}(a^2 - r^2).$$

Continuity requires $B = Aa^5$, hence

$$\Phi_- = \frac{A}{2} r^2 (3\cos^2\vartheta - 1) + \frac{\rho_2 + \rho_3}{6\epsilon}(a^2 - r^2)$$
$$\Phi_+ = \frac{a^5 A}{2r^3}(3\cos^2\vartheta - 1).$$

Using (32), the r.h.s. of (34) may be written in the form

$$\sigma_2 + \sigma_3 = \frac{1}{9}(2\epsilon + \epsilon_0) a\mu_0 M\Omega[2 - (3\cos^2\vartheta - 1)].$$

Then (34) becomes

$$\begin{aligned}\epsilon_0 \cdot \frac{3}{2} aA(3\cos^2\vartheta - 1) + \epsilon a A(3\cos^2\vartheta - 1) - \frac{1}{3}(\rho_2 + \rho_3)a = \\ = \frac{1}{9}(2\epsilon + \epsilon_0) a\mu_0 M\Omega[2 - (3\cos^2\vartheta - 1)]\end{aligned} \tag{42}$$

As it follows from (31) the terms which do not contain $(3\cos^2\vartheta - 1)$ cancel and the remaining equation determines the value of A:

$$A = -\frac{2}{9} \cdot \frac{2\epsilon + \epsilon_0}{2\epsilon + 3\epsilon_0} \mu_0 M\Omega.$$

From this (35) follows. The cancellation of the terms which do not contain $P_2(\cos\vartheta)$ is the consequence of the fact that, according to (28) and (29), the volume and surface charges compensate to zero separately for all three types of polarization charges.