Online Appendix to the book
Basic Relativity — An Introductory Essay

Contents

This Appendix contains those supplementary remarks which were left out of the book owing to their advanced character.

1 On the prehistory of the relativistic mass

At the end of the 19. century J. J. Thomson, H. A. Lorentz, M. Abraham and others began to apply Maxwellian electrodynamics to charged constituents of matter called *electrons*. In those times this term denoted microscopic particles of either sign of charge. They recognized that in the electrostatic field of electrons at rest a certain amount of field energy W_e is stored whose magnitude depends on how the charge is distributed within the volume of the particle. When the electron moves with the velocity V it is surrounded by both electric and magnetic field which contains, beside field energy, field momentum p_{em} too. The *total momentum* of the particle is, therefore, given by the formula

$$p_{total} = mV + p_{em}. \tag{1.1}$$

The value of p_{em} could be calculated once the structure of the particle had been specified. The calculation was confined to a single inertial frame at rest in the ether and so uncertainties concerning field transformation between different inertial frames presented no obstacle. For a 'metallic' ball with only surface charge on it the calculation lead to

$$p_{em} = \frac{4}{3} \cdot \frac{V}{\sqrt{1 - V^2/c^2}} \cdot \frac{W_e}{c^2}. \tag{1.2}$$

This field momentum is the higher the faster the electron moves. Therefore, at high velocities it leads to the suppression of the acceleration since an increasingly larger fraction of the thrust of the force is spent on the augmentation of the field momentum instead of adding to particle's speed. If we insert (1.2) into (1.1) we obtain

$$p_{total} = \left(m + \frac{4}{3} \cdot \frac{1}{\sqrt{1 - V^2/c^2}} \cdot \frac{W_e}{c^2} \right) \cdot V. \tag{1.3}$$

The second term in the brackets plays formally the role of a mass and the name *electromagnetic mass* has indeed been given to it. Then

$$m_{total} = m + m_{em}. \tag{1.4}$$

This identification is, however, an unnecessary and even a rather confusing one because field momentum and mass are conceptually very different notions. It is owing to this confusing notation that acceleration suppression due to the augmentation of the field momentum became known under the misleading term of a mass depending on velocity.

The electromagnetic theory of mass consisted in the further assumption that m is in fact equal to zero and the total mass is of purely electromagnetic origin:

$$m_{total} = \frac{4}{3} \cdot \frac{1}{\sqrt{1 - V^2/c^2}} \cdot \frac{W_e}{c^2}. \tag{1.5}$$

The experiments of W. Kaufmann were conceived to verify these ideas.

It was, however, pointed out by H. Poincaré that the electromagnetic theory of mass is essentially incomplete since the stability of the electron cannot be ensured by electrostatic forces alone and any additional field of force (*Poincaré stresses*) required by stability contributes also to the field energy and momentum.

The situation has been clarified only in relativity theory. Local energy and momentum conservation requires the divergence of the total energy-momentum tensor to vanish. For a classical electron the electromagnetic part of this tensor has a divergence different from zero and contribution from Poincaré-stresses is indeed necessary to annul it. The vanishing of this divergence, on the other hand, ensures that the total field energy and momentum constitute a four vector of the same nature as the four-momentum of a particle. This circumstance permits one to interprete the energy and the momentum of the classical electron as originating from its electromagnetic field and Poincaré-stresses. The four-vector character of the energy-momentum leads, instead of (1.2), to the formula

$$p_{field} = \frac{V}{\sqrt{1 - V^2/c^2}} \cdot \frac{E_0}{c^2} \tag{1.6}$$

where E_0 is the rest energy originating from both the electromagnetic field and Poincaré-stresses. The cancellation of the factor $4/3$ can be attributed to the contribution of the Poincaré-stresses.

But the appeal of this interpretation is considerably weakened by the fact that the right hand side of (1.6) describes in fact the momentum of both charged and neutral particles since, according to relativity theory, it is the direct consequence of the basic principles of this theory without any reference to the object's structure (see Sections 1.7-1.8). It is only radiation reaction which discriminates between the acceleration of charged and neutral particles.

Returning to the electromagnetic theory of mass we see that according to (1.5) the mass m_0 of a particle at rest is equal to $\frac{4}{3} \cdot \frac{W_e}{c^2}$. Based on this relation it is sometimes argued that mass-energy relation has already been foreseen before relativity theory. This supposition is, however, completely untenable. As it is seen from its derivation in Section 1.10 the mass-energy relation $E_0 = mc^2$ is the consequence of the equivalence of the inertial frames (while before the birth of relativity theory the different inertial frames were in fact distinguished from each other by their motion in the ether). The general validity of it is ensured by this equivalence. One of its most astonishing consequence is the explanation of the origin of the energy release in radioactive decay which was pointed out by Einstein already in its very first paper on the issue. The thermal effect of radioactivity was known a decade before Einstein's paper but nobody even dreamt of relating it to the formula $\frac{4}{3} \cdot \frac{W_e}{c^3}$. As a matter of fact the mass-energy relation $E_0 = mc^2$ has no forerunner (proper prehistory) at all.

2 Gravitational red shift of photons

In Section 3.9 the gravitational red shift formula for light pulses was derived. The derivation was based on the time dilation of the period between two subsequent pulses and it is evidently applicable to classical electromagnetic waves too. The red shift of photons, however, must be based on their energy since the phase of the electromagnetic waves associated with them is indefinite.

As in Section 3.9 consider again first the transverse Doppler-effect in special relativity. The emitter and the receiver are rotating with a constant angular velocity Ω on concentric circles at a distance r_e and r_r from the center of rotation. Then their speeds are equal to $V_e = r_e\Omega$ and $V_r = r_r\Omega$. Let us attach a common rotating cylindrical coordinate system $\mathcal{K}$ to them in which they are at rest. The metric in it is given by the formula

$$ds^2 = \left(1 - \frac{r^2\Omega^2}{c^2}\right) d(ct)^2 - 2r^2\Omega\, dt \cdot d\varphi - dr^2 - r^2\, d\varphi^2 - dz^2. \qquad (2.1)$$

The mass of the local laboratory $\mathcal{R}_e$ where the emission takes place is denoted by M_e. After the emission of the photon it becomes equal to $(M_e - \Delta M_e)$. Then the energy loss of $\mathcal{R}_e$ in $\mathcal{K}$ is equal to

$$\Delta E_e = \Delta M_e \cdot c^2 \cdot \sqrt{1 - {V_e}^2/c^2}. \qquad (2.2)$$

To prove this formula we notice that the energy of a body of mass m is equal to mc-times the *covariant* null-component V_0 of its four velocity $V^i = \frac{dx^i}{d\tau}$ $(i = 0, 1, 2, 3)$. The energy of the body at rest is, therefore, equal to

$$mcV_0 = mc\, g_{00} V^0 = mc\, g_{00} \frac{dx^0}{d\tau} = mc^2 g_{00} \frac{dt}{d\tau} = mc^2 \sqrt{g_{00}} \qquad (2.3)$$

since for a body at rest $d\tau = \sqrt{g_{00}} \cdot dt$. This formula in conjunction with the metric (2.1) leads indeed to (2.2)

Consider the coordinate system $\mathcal{K}_e$ attached to $\mathcal{R}_e$. Its spatial part coincides with the local piece of $\mathcal{K}$ but the coordinate time in it is equal to the proper time of the emitter since it is measured by clocks fixed in the laboratory $\mathcal{R}_e$. Therefore, in $\mathcal{K}_e$ the component g_{00} of the metric tensor is equal to 1 and by (2.3) the energy loss in it is $\Delta M_e \cdot c^2$. Then the frequency associated with the photon[1] is equal to $\nu_e = \frac{1}{h}\Delta M_e \cdot c^2$. Substituting ΔM_e from this formula into (2.2) we obtain

$$\Delta E_e = h\nu_e\sqrt{1 - {V_e}^2/c^2}.$$

For the energy gain of the receiver in $\mathcal{K}$ we obtain similarly

$$\Delta E_r = h\nu_r\sqrt{1 - {V_r}^2/c^2}.$$

Notice that the frequencies ν_e and ν_r in these formulae are measured in the proper time of the emitter and the receiver respectively.

Now the metric (2.1) is independent of t and so the energy of the photon is conserved during its travel from the emitter to the receiver. Hence $\Delta E_e = \Delta E_r$ and for the frequencies associated with the photon in the emitter's and the receiver's local rest frames $\mathcal{R}_e$ and $\mathcal{R}_r$ we obtain the same transverse Doppler-shift formula

$$\frac{\nu_r}{\nu_e} = \frac{\sqrt{1 - {V_e}^2/c^2}}{\sqrt{1 - {V_r}^2/c^2}}$$

which was found in Section 3.9 for classical radiation.

The gravitation red-shift formula

$$\frac{\nu_r}{\nu_e} = \frac{\sqrt{1 - 2\Phi_e/c^2}}{\sqrt{1 - 2\Phi_r/c^2}}$$

of Section 3.9 can be verified for photons by a completely analogous reasoning if instead of (2.1) the Schwarzschild-metric

$$ds^2 = \left(1 - 2\Phi/c^2\right) d(ct)^2 - \frac{1}{1 - 2\Phi/c^2} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta \cdot d\varphi^2)$$

is applied.

[1] $E = h\nu$ is the zeroth covariant component of the four-vector relation $\mathbf{p} = \hbar\mathbf{k}$ valid in *any* coordinates, in particular, in the coordinate systems $\mathcal{K}_e$ and $\mathcal{K}_r$ attached to the noninertial frames $\mathcal{R}_e$ and $\mathcal{R}_r$.